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MATHÉMATIQUE

Propriété limite du potentiel spécial de simple couche d'un système parabolique d'équations

par
H. MILICER-GRUŻEWSKA

Présenté par A. ZYGMUND, le 27 mars, 1961

Introduction. Certaines hypothèses étant admises, la limite du potentiel généralisé de simple couche, défini par Pogorzelski [1], est le potentiel généralisé du système élliptique d'équations limites [2]. Le nombre n de dimension de l'espace E_n, où le système fut défini, était plus grand que le rang M de ce système ($n > M$) (voir aussi [3]). Dans la présente communication le cas $n \leqslant M$ est étudié. Mais dans ce cas, on le sait, [4], il est inutile de chercher la limite du potentiel, dont il fut question. On a donc construit, en joignant les idées de Eidelman [5] et de Pogorzelski [1], la solution fondamentale spéciale du système parabolique d'équations. Cette solution permet de définir aussi bien l'intégrale généralisée de Poisson-Weierstrass spéciale que les potentiels spéciaux — celui de simple couche ou spatial — des densités convenables, l'intégrale et les potentiels étant liés avec le système parabolique.

Les hypothèses du travail [2], appliquées aux nouvelles définitions seront légèrement changées, mais leur essentiel sera maintenu. Dans les hypothèses, précisées au par. **1**, on *démontre que la limite du potentiel spécial de simple couche, lié avec le système parabolique d'équations est le potentiel spécial de simple couche du système élliptique d'équations limites.*

1. Soit E_n l'espace euclidien à n dimensions:

$$X = (x_1, ..., x_n); \quad Y = (y_1, ..., y_n); \quad Z = (z_1, ..., z_n)$$

t — le temps non-négatif. Soient: Ω un domaine de E_n; $\bar{\Omega}$ — la ferméture de Ω; Ω' une région de E_n contenant $\bar{\Omega}$; $c, C, c_0, C^0, ...$ les constantes positives (on les écrira aussi: const. ou $\mathcal{C}^{\text{te}}$). Soit $M \geqslant n$ l'ordre du système parabolique d'équations linéaires, normales, aux coefficients höldériens et dérivées partielles; $k_1, ..., k_n$ — les composants entiers non-négatifs de chaque entier non-négatif $k \leqslant M$. On a alors $0 \leqslant k_i \leqslant k$; $k_1 + ... + k_n = k$; $0 \leqslant k \leqslant M \geqslant n$. On posera: $(k_1, ..., k_n) = (k)$ et on écrira les coefficients du système et les dérivées de la solution inconnue $(u_1, ..., u_N) = (u)$ comme suit:

$$A_{\alpha\beta}^{(k)}(X, t) = A_{\alpha\beta}^{k_1, ..., k_n}(X, t); \qquad D^k(u) = \partial^{k_1 + ... + k_n} u / \partial x_1^{k_1}, ..., \partial x_n^{k_n}.$$

Le système d'équations sera ici exprimé, comme dans l'ouvrage [2], par

$$(1,1)\quad \hat{\varphi}^{(a)}(u) = \sum_{1\leqslant j\leqslant N}^{0\leqslant k\leqslant M} A_{aj}^{(k)}(X,t)\,D^k(u_j) - \partial u_\alpha/\partial t \equiv 0,\ a=1,\dots,N,\ X\in\Omega,\ t>0,$$

la somme (1,1) étant répandue sur toutes les suites (k).

En désignant par p le nombre des composants de la somme (1,1) et en posant $(a_1 b_1 + \dots + a_p b_p) = (a\cdot b)_1^p$, on peut écrire le système (1,1) sous une forme plus abrégée:

$$(1',1)\quad \hat{\varphi}^{(a)}(u) = (A_\alpha(X,t)\cdot D(u))_1^p - \partial u_\alpha/\partial t = 0,\quad a=1,\dots,N,\quad X\in\Omega,\quad t>0.$$

Les hypothèses sur les coefficients du système (1,1) sont les suivantes:

HYPOTHÈSE I. *Les coefficients du système* (1,1) *sont continus et bornés dans la région* Ω' *et pour* $t\geqslant 0$; *ils sont soumis aux conditions de Hölder*:

$$(2,1)\qquad |A_{aj}^{(k)}(X,t) - A_{aj}^{(k)}(Y,\tau)| \leqslant \begin{cases} C_0\,[|XY|^h + |t-\tau|^{h'}], & k=M \\ C^0\,|XY|^h, \quad k<M, & t=\tau \end{cases}$$

où $|XY|$ *désigne la distance entre les points* X *et* Y, h *et* h' *sont constants et* $0<h\leqslant 1$; $0<h'\leqslant 1$. *Ecrivons*: $h_1 = \min(h, Mh')$.

On suppose aussi que les limites:

$$(2'',1)\qquad \lim_{t\to\infty} A_{aj}^{(k)}(X,t) = \hat{A}_{aj}^{(k)}(X),\quad X\in\Omega',\quad a,j=1,\dots,N$$

existent uniformément dans Ω' *pour chaque suite* (k) *définie plus haut.*

HYPOTHÈSE II. *Le système* (1,1) *est uniformément parabolique par rapport à la variable* t *dans la région* Ω' *d'après Petrovsky*, [6].

Appliquons à présent la méthode d'Eidelman de l'article [5] en développant la quasi-solution bien connue $W_{\alpha\beta}^{Z,\xi}(X,t;\ Y,\tau)$ du systéme (1,1) (voir [1], 2.) en série de Taylor au point $A=(a_1,\dots,a_n)$, $a_1^2+\dots+a_n^2 = |A|^2>0$:

$$(3,1)\qquad W_{\alpha\beta}^{Z,\xi} = \sum_{p=0}^{\infty} {}_AQ_p^{Z,\xi}(X,t;\ Y,\tau) = \sum_{p=0}^{\infty} {}_AQ_p^{Z,\xi},$$

où on a posé

$$(4,1)\qquad {}_AQ_p^{Z,\xi} = \frac{1}{p!}(\xi_1\,\partial/\partial x_1 + \dots + \xi_n\,\partial/\partial x_n)^{(p)}\,W_{\alpha\beta}^{Z,\xi}(X,t;\ Y,\tau)\big|_{X-Y=A}$$

en écrivant: $\xi_i = x_i - y_i - a_i$.

Désignons:

$$(5,1)\qquad {}_AP_{\alpha\beta}^{Z,\xi} = {}_AP_{\alpha\beta}^{Z,\xi}(X,t;\ Y,\tau) = \sum_{p=0}^{m-n} {}_AQ_p^{Z,\xi}.$$

Définition (1,1). La quasi- solution spéciale du système (1,1) est définie par la formule:

$$(6,1)\qquad {}_AW_{\alpha\beta}^{Z,\xi} = {}_AW_{\alpha\beta}^{Z,\xi}(X,t;\ Y,\tau) = W_{\alpha\beta}^{Z,\xi} - {}_AP_{\alpha\beta}^{Z,\xi},\quad \alpha,\beta=1,\dots,N$$

et le noyau spécial de ce système par:

(7,1) $$ {}_AN_{\alpha\beta} = {}_AN_{\alpha\beta}(X, t;\; Y, \tau) = \hat{\psi}^{(\alpha)}({}_AW_{\alpha\beta}^{Y,\tau}), \quad \alpha, \beta = 1, \ldots, N. $$

Définition (2,1). La solution fondamentale spéciale du système (1,1) est définie comme il suit:

(8,1) $$ {}_A\Gamma_{\alpha\beta} = {}_A\Gamma_{\alpha\beta}(X, t;\; Y, \tau) = $$
$$ = {}_AW_{\alpha\beta}^{Y,\tau} + \int_\tau^t \int_{\Omega'} \sum_{\gamma=1}^{N} {}_AW_{\alpha\gamma}^{\Pi,\xi}(X, t;\; \Pi, \xi)\, {}_A\Phi_{\gamma\beta}(\Pi, \xi;\; Y, \tau)\, d\Pi\, d\xi, $$

où

(9,1) $$ {}_A\Phi_{\alpha\beta}(X, t;\; Y, \tau) = {}_A\Phi_{\alpha\beta}, \quad \alpha, \beta = 1, \ldots, N $$

est la solution du système d'équations intégrales:

(10,1) $$ {}_A\Phi_{\alpha\beta} = {}_AN_{\alpha\beta} + \int_\tau^t \int_{\Omega'} \sum_{\gamma=1}^{N} {}_AN_{\alpha\gamma}(X, t; \Pi, \xi)\, \Phi_{\gamma\beta}(\Pi, \xi; Y, \tau)\, d\Pi\, d\xi, \quad \alpha = 1, \ldots, N. $$

On construit la matrice des fonctions dominantes en posant: $t - \tau = \theta$ et:

(11,1) $$ \dot{N}_{\alpha\beta} = \dot{N}_{\alpha\beta}(X, Y, \theta) = \begin{cases} \mathscr{C}^{te}\, \theta^{-(1+1/M)}, & \theta > 1,\; X, Y, \in \Omega' \\ \mathscr{C}^{te}\, \theta^{-[1+(n-h)/M]} \exp\left[-c\left(|XY|/\theta^{1/M}\right)^q\right], & \end{cases} $$

où les constantes $\mathscr{C}^{te}$ et $c > 0$ sont ajustées convenablement aux Hypothèses I et II.

Hypothèse III. *On suppose le domaine Ω' adapté aux inégalités*:

(12,1) $$ \int_0^\theta \int_{\Omega'} \sum_{\beta=1}^{N} \dot{N}_{\alpha\beta}(X, Y, \xi)\, dY\, d\xi \leqslant b < 1 $$

et

(12',1) $$ \int_0^\theta \int_{\Omega'} \sum_{\beta=1}^{N} \dot{N}_{\beta\alpha}(X, Y, \xi)\, dX\, d\xi \leqslant b < 1 $$

$$ \alpha = 1, \ldots, N; \quad X \in \Omega; \quad Y \in \Omega'; \quad 0 \leqslant \theta \leqslant \infty. $$

Hypothèse IV. *La fonction* $\{\varphi_\alpha(Q, t)\}_{\alpha=1,\ldots,N} = \varphi(Q, t)$ *est définie, continue, intégrable et bornée sur la surface S du domaine Ω pour chaque valeur du temps $t \geqslant 0$. On suppose qu'on a uniformément sur S*:

(13,1) $$ \lim_{t\to\infty} \varphi_\alpha(Q, t) = \hat{\varphi}_\alpha(Q), \quad Q \in S, \; \alpha = 1, \ldots, N. $$

Le potentiel généralisé special (A) de simple couche rélativement au système (1,1) de la densité $\varphi(Q, t)$ est le suivant:

(14,1) $$ {}_AU_\alpha(X, t) = \int_0^t \int_S \sum_{\gamma=1}^{N} {}_A\Gamma_{\alpha\gamma}(X, t;\; Q, \tau)\, \varphi_\gamma(Q, \tau)\, dQ\, d\tau, \quad Q \in S. $$

2. On fait intervenir les définitions suivantes:

le système d'équations:

$$(1,2)\qquad \hat{\psi}^{(\alpha)}(\hat{u}) = (\hat{A}_\alpha(X)\cdot D(\hat{u}))_1^p - \partial \hat{u}_\alpha/\partial t \equiv 0\,, \qquad \alpha = 1, \ldots, N,$$

qui est parabolique comme le système (1,1);

les matrices: des quasi-solutions spéciales, des solutions fondamentales spéciales et des noyaux spéciaux, correspondants au système (1,2):

$$(2,2)\qquad \overset{A}{W}{}^{Z}_{\alpha\beta} = \overset{A}{W}{}^{Z}_{\alpha\beta}(X, Y, \theta),$$

$$(2',2)\qquad \overset{A}{\Gamma}_{\alpha\beta} = \overset{A}{\Gamma}_{\alpha\beta}(X, Y, \theta),$$

$$(2'',2)\qquad \overset{A}{N}_{\alpha\beta} = \overset{A}{N}_{\alpha\beta}(X, Y, \theta).$$

On fait intervenir aussi:

les intégrales:

$$(3,2)\qquad \overset{A}{w}{}^{Z}_{\alpha\beta} = \overset{A}{w}{}^{Z}_{\alpha\beta}(X, Y) = \int\limits_0^\infty \overset{A}{W}{}^{Z}_{\alpha\beta}(X, Y, \theta)\, d\theta,$$

$$(4,2)\qquad \overset{A}{n}_{\alpha\beta} = \overset{A}{n}_{\alpha\beta}(X, Y) = \int\limits_0^\infty \overset{A}{N}_{\alpha\beta}(X, Y, \theta)\, d\theta;$$

les itérations $\overset{A}{n}{}^{(\nu)}_{\alpha\beta}$ des fonctions $\overset{A}{n}_{\alpha\beta}$ et leurs somme infinie:

$$(5,2)\qquad \overset{A}{m}_{\alpha\beta} = \overset{A}{m}_{\alpha\beta}(X, Y) = \sum_{\nu=0}^{\infty} \overset{A}{n}{}^{(\nu)}_{\alpha\beta}(X, Y), \qquad \overset{A}{n}{}^{(0)}_{\alpha\beta} = \overset{A}{n}_{\alpha\beta}\,.$$

Il résulte des hypothèses du par. **1** que toutes ces définitions sont légitimes et que la fonction $\overset{A}{m}_{\alpha\beta}$ est égale à la solution du système des équations intégrales:

$$(6,2)\qquad \overset{A}{\varphi}_{\alpha\beta}(X, Y) = \overset{A}{n}_{\alpha\beta}(X, Y) + \int\limits_{\Omega'} \sum_{\nu=1}^{N} \overset{A}{n}_{\alpha\gamma}(X, \Pi)\, \overset{A}{\varphi}_{\gamma\beta}(\Pi, Y)\, d\Pi,$$

$$\alpha, \beta = 1, \ldots, N; \qquad \beta = \text{const}; \qquad X \in \Omega,\ Y \in \Omega'; \qquad X \neq Y.$$

Ainsi on a:

$$(7,2)\qquad \overset{A}{m}_{\alpha\beta}(X, Y) = \overset{A}{\varphi}_{\alpha\beta}(X, Y)\,.$$

On définit la matrice des solutions fondamentales spéciales

$$(8,2)\qquad \overset{A}{\Gamma}_{\alpha\beta}(X, Y) = \overset{A}{w}{}^{Y}_{\alpha\beta}(X, Y) + \int\limits_{\Omega} \sum_{\gamma=1}^{N} \overset{A}{w}{}^{\Pi}_{\alpha\gamma}(X, \Pi)\, \overset{A}{\varphi}_{\gamma\beta}(\Pi, Y)\, d\Pi$$

et celle du systéme élliptique d'équations:

$$(9,2)\qquad \psi^{(\alpha)}(u) = (\hat{A}_\alpha(X)\, D(u))_1^p \equiv 0\,.$$

Le potentiel de simple couche généralisé spécial de ce système élliptique est:

$$(10,2)\qquad \overset{A}{u}_\alpha(X)=\int_S \sum_{\gamma=1}^{N} \overset{A}{\Gamma}_{\alpha\gamma}(X,Q)\,\hat{\varphi}_\gamma(Q)\,dQ,\qquad X\in\Omega,\ \alpha=1,\dots,N.$$

Donc le théorème suivant est vrai

THÉORÈME. *Les Hypothèses* I—IV *étant admises, l'ordre du système* $M\geqslant n$, *le temps* $t\to\infty$, *la limite du potentiel généralisé spécial,* $(A,\ |A|>0)$, *de simple couche,* (10,1), *du système parabolique* (1,1) *est le potentiel généralisé spécial* (A) *de simple couche,* (10,2), *du système élliptique limite* (9,2) *et de densité limite.*

On a notamment:

$$(11,2)\qquad \lim_{t\to\infty} {}_AU_\alpha(X,t)=\overset{A}{u}_\alpha(X),\ X\in\Omega.$$

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OUVRAGES CITÉS

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MATHEMATICS

Determinant Systems for Generalized Fredholm Operators

by

A. BURACZEWSKI

Presented by E. MARCZEWSKI on April 13, 1961

1. Introduction

Let Ξ and X be two linear spaces over the same real or complex field $\mathfrak{F}$. The letters ξ, η, ζ (with indices) denote elements of Ξ, x, y, z — elements of X, and a, b, c — scalars of $\mathfrak{F}$. Every mapping into $\mathfrak{F}$ will be called functional.

We suppose that Ξ and X are *conjugate* [4], i.e. there exists a bilinear functional on $\Xi \times X$ whose value at the point (ξ, x) is denoted by ξx and which satisfies two additional conditions:

(a) if $\xi x = 0$ for every $\xi \in \Xi$, then $x = 0$;

(a′) if $\xi x = 0$ for every $x \in X$, then $\xi = 0$.

Let $\mathfrak{A}$ be a class of all endomorphisms A in X such that the following condition is satisfied

(b) for every fixed $x \in X$ there exists a $y \in X$ such that $\xi(Ax) = \xi y$ for every $\xi \in \Xi$.

Every endomorphism $A \in \mathfrak{A}$ induces an adjoint one in Ξ which will also be denoted by A and whose value at the point ξ is denoted by ξA. It satisfies the condition

(b′) for every fixed $\xi \in \Xi$ there exists an $\eta \in \Xi$ such that $(\xi A)x = \eta x$ for every $x \in X$.

Endomorphisms $A \in \mathfrak{A}$ can be interpreted as bilinear functionals on $\Xi \times X$ as follows: $\xi A x = \xi(Ax) = (\xi A)x$. It is obvious that $\mathfrak{A}$ is a ring. For fixed ξ_0, x_0, let $x_0 \cdot \xi_0$ denote the *one-dimensional* operator $K \in \mathfrak{A}$ such that $Kx = x_0 \cdot \xi_0 x$. Instead of writing ξx, we can write $\xi I x$, where I is the identity operator. An operator (bilinear functional) $B \in \mathfrak{A}$ is said to be a *quasi-inverse* of $A \in \mathfrak{A}$ [4], if $ABA = A$, $BAB = B$. An operator $A \in \mathfrak{A}$ is said to be a *generalized Fredholm* operator if: the equation $Ax = 0$ has exactly n linearly independent solutions $z_1, \ldots, z_n$; the equation $\xi A = 0$ has exactly m linearly independent solutions $\zeta_1, \ldots, \zeta_m$; the equation $Ax = x_0$ has a solution x if and only if $\zeta_i x_0 = 0$ for $i = 1, \ldots, m$; the equation $\xi A = \xi_0$ has a solution ξ if and only if $\xi_0 z_i = 0$ for $i = 1, \ldots, n$. The integers $r(A) = \min(m, n)$, $d(A) = n - m$ will be called the *order* and the *defect* of A, respectively. If $d(A) = 0$, then A is said to be called a *Fredholm operator*.

The following lemmas hold [1]:

(i) *If A is a generalized Fredholm operator and $C \in \mathfrak{A}$ has the inverse $C^{-1} \in \mathfrak{A}$, then so are CA and AC, and $d(CA) = d(AC) = d(A)$, $r(CA) = r(AC) = r(A)$.*

(ii) *Every generalized Fredholm operator $A \in \mathfrak{A}$ has a quasi-inverse $B \in \mathfrak{A}$ and $d(B) = -d(A)$, $r(B) = r(A)$.*

(iii) *$A \in \mathfrak{A}$ is a generalized Fredholm operator if and only if there exist $B_1, B_2 \in \mathfrak{A}$ and a finitely dimensional $K_1, K_2 \in \mathfrak{A}$ such that $B_1 A = I - K_1$, $AB_2 = I - K_2$.*

(iv) *If $A_1, A_2 \in \mathfrak{A}$ are generalized Fredholm operators, then so is $A_1 A_2$ and $d(A_1 A_2) = d(A_1) + d(A_2)$.*

(v) *If A is a generalized Fredholm operator, then A can be represented in the form $A = S + K$, where $S \in \mathfrak{A}$ is a generalized Fredholm operator such that $r(S) = 0$, $d(S) = d(A)$.*

Moreover, if $U \in \mathfrak{A}$ is a quasi-inverse of S, then $r(I+UK) = r(I+KU) = r(A)$.

2. Definition of the determinant system

Using the terminology of R. Sikorski [4] we shall understand by a *determinant system* for an operator $A \in \mathfrak{A}$ every infinite sequence $D_0, D_1, \ldots$ such that:

(d_1) D_n is a $2n+l$-linear functional on $\Xi^{n+l} \times X^n$; the value at the point $(\xi_1, \ldots, \xi_{n+l}, x_1, \ldots, x_n)$ will be denoted by $D_n \begin{pmatrix} \xi_1, \ldots, \xi_{n+l} \\ x_1, \ldots, x_n \end{pmatrix}$;

(d_2) $D_n \begin{pmatrix} \xi_1, \ldots, \xi_{n+l} \\ x_1, \ldots, x_n \end{pmatrix}$ is skew symetric in $\xi_1, \ldots, \xi_{n+l}$ and in $x_1, \ldots, x_n$;

(d_3) if $D_n \begin{pmatrix} \xi_1, \ldots, \xi_{n+l} \\ x_1, \ldots, x_n \end{pmatrix}$ is interpreted as a function of ξ_i only, then there exists an element $z_i \in X$ such that

$$D_n \begin{pmatrix} \xi_1, \ldots, \xi_{n+l} \\ x_1, \ldots, x_n \end{pmatrix} = \xi_i z_i \text{ for every } \xi_i \in \Xi;$$

if $D_n \begin{pmatrix} \xi_1, \ldots, \xi_{n+l} \\ x_1, \ldots, x_n \end{pmatrix}$ is interpreted as a function of x_j only, then there exists an element $\zeta_j \in \Xi$ such that

$$D_n \begin{pmatrix} \xi_1, \ldots, \xi_{n+l} \\ x_1, \ldots, x_n \end{pmatrix} = \zeta_j x_j \text{ for every } x_j \in X;$$

(d_4) there exists an integer $r \geqslant 0$ such that D_r does not vanish identically;

(d_5) the following identities hold for $n = 0, 1, \ldots$:

$$D_{n+1} \begin{pmatrix} \xi_0 A, \xi_1, \ldots, \xi_{n+l} \\ x_0, \quad x_1, \ldots, x_n \end{pmatrix} = \sum_{i=0}^{n} (-1)^i \xi_0 x_i D_n \begin{pmatrix} \xi_1, \ldots, \quad \ldots\ldots\ldots \quad \xi_{n+l} \\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \end{pmatrix},$$

$$D_{n+1} \begin{pmatrix} \xi_0, \quad \xi_1, \ldots, \xi_{n+l} \\ Ax_0, x_1, \ldots, x_n \end{pmatrix} = \sum_{i=0}^{n+l} (-1)^i \xi_i x_0 D_n \begin{pmatrix} \xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_{n+l} \\ x_1, \ldots, \quad \ldots\ldots\ldots \quad x_n \end{pmatrix}.$$

The definition of the determinant system, in the case when the number of x_i is larger than that of ξ_i, is analogous. The least integer $r = r(D_n)$ such that D_r does not vanish identically and the difference $d(D_n)$ between the numbers of x_i and ξ_i in D_n are called the *order* and the *defect* of the determinant system, respectively.

In the sequel, to avoid the duality of formulation, we shall only consider operators with positive defect.

(vi) *If $S \in \mathfrak{A}$ is a generalized Fredholm operator of the order zero, $U \in \mathfrak{A}$ is a quasi-inverse of S, and $s_1, \ldots, s_{d(S)}$ are all linearly independent solutions of the equation $Sx = 0$, then the sequence $\{\Theta_n\}$ defined as follows*

$$\Theta_n\begin{pmatrix}\xi_1, \ldots, \xi_p\\ x_1, \ldots, x_n\end{pmatrix} = \begin{vmatrix} \xi_1 Ux_1, \ldots, \xi_1 Ux_n, \xi_1 s_1, \ldots, \xi_1 s_{d(S)}\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \xi_p Ux_1, \ldots, \xi_p Ux_n, \xi_p s_1, \ldots, \xi_p s_{d(S)}\end{vmatrix}$$

where $p = n+d(S)$ $(n = 0, 1, \ldots)$, is a determinant system for S.

3. Fundamental theorems

(vii) *If $A \in \mathfrak{A}$ has a determinant system $\{D_n\}$, then A is a generalized Fredholm operator such that $r(A) = r(D_n)$, $d(A) = d(D_n)$.*

More exactly: *If $r = r(D_n)$, $l = d(D_n) \geqslant 0$, and $\eta_1, \ldots, \eta_{r+l} \in \Xi$, $y_1, \ldots, y_r \in X$ are fixed elements such that $\delta = D_r\begin{pmatrix}\eta_1, \ldots, \eta_{r+l}\\ y_1, \ldots, y_r\end{pmatrix} \neq 0$, then there exist elements $\zeta_1, \ldots, \zeta_r \in \Xi$, $z_1, \ldots, z_{r+l} \in X$ and a quasi-inverse $B \in \mathfrak{A}$ of A such that, for all $\xi \in \Xi$, $x \in X$*

$$\zeta_i x = \delta^{-1} D_r\begin{pmatrix}\eta, \ldots, \ldots\ldots\ldots \eta_{r+l}\\ y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_r\end{pmatrix},$$

$$\xi z_i = \delta^{-1} D_r\begin{pmatrix}\eta_1, \ldots, \eta_{i-1}, \xi, \eta_{i+1}, \ldots, \eta_{r+l}\\ y_1, \ldots, \ldots\ldots\ldots y_r\end{pmatrix},$$

$$\xi Bx = \delta^{-1} D_{r+1}\begin{pmatrix}\xi, \eta_1, \ldots, \eta_{r+l}\\ x, y_1, \ldots, y_r\end{pmatrix}.$$

Every solution ξ of $\xi A = 0$ is a linear combination of $\zeta_1, \ldots, \zeta_r$, and every solution x of $Ax = 0$ is a linear combination of $z_1, \ldots, z_{r+l}$. The equation $\xi A = \xi_0$ has a solution ξ if and only if $\xi_0 z_i = 0$ for $i = 1, \ldots, r+l$ Then the general form of the solution ξ is $\xi = c_1 \zeta_1 + \ldots + c_r \zeta_r + \xi_0 B$.

The equation $Ax = x_0$ has a solution x if and only if $\zeta_i x_0 = 0$ for $i = 1, \ldots, r$. Then the general form of the solution x is $x = c_1 z_1 + \ldots c_{r+l} z_{r+l} + Bx_0$.

(viii) *If $A \in \mathfrak{A}$ is a generalized Fredholm operator, then A has a determinant system $\{D_n\}$, and $r(D_n) = r(A)$, $d(D_n) = d(A)$. This system is determined by A uniquely up to a scalar factor $\neq 0$.*

(ix) *Let $A = S+K$ (see (v)) be such that $r(A) = r$, $d(A) = l \geqslant 0$ and let $s_1, \ldots, s_l$ be linearly by independent solutions of $Sx = 0$. Let U be a quasi-inverse of S and let $\{\bar{D}_n\}$ be a determinant system for the Fredholm operator $I+UK$.*

The sequence $D_0, D_1, \ldots$ *defined by the formula*

$$D_n\begin{pmatrix}\xi_1, \ldots, \xi_{n+l}\\ x_1, \ldots, x_n\end{pmatrix} = D_n\begin{pmatrix}\xi_1, \ldots, & \ldots & \xi_{n+l}\\ Ux_1, \ldots, Ux_n, s_1, \ldots, s_l\end{pmatrix} \quad n = 0, 1, \ldots$$

is a determinant system for $S+K$ *such that* $r(D_n) = r(\bar{D}_n)$.

Let $\eta_1, \ldots, \eta_r \in \Xi$ *and* $y_1, \ldots, y_r \in X$ *be such that* $\delta = \bar{D}_r\begin{pmatrix}\eta_1, \ldots, \eta_r\\ y_1, \ldots, y_r\end{pmatrix} \neq 0$.

The elements $\zeta_1, \ldots, \zeta_r \in \Xi$ *and* $z_1, \ldots, z_{r+l} \in X$, *such that*

$$\zeta_i x = \frac{1}{\delta}\bar{D}_r\begin{pmatrix}\eta_1, \ldots, & \ldots & \eta_r\\ y_1, \ldots, y_{i1}, Ux, y_{i1}, \ldots, y_r\end{pmatrix} \quad \text{for every } x \in X,$$

$$\xi z_i = \frac{1}{\delta}\bar{D}_r\begin{pmatrix}\eta_1, \ldots, \eta_{i-1}, \xi, \eta_{i+1}, \ldots, \eta_r\\ y_1, \ldots, & \ldots & y_r\end{pmatrix} \quad \text{for every } \xi \in \Xi,$$

$$\xi z_{r+j} = \frac{1}{\delta}\bar{D}_{r+1}\begin{pmatrix}\xi, \eta_1, \ldots, \eta_r\\ s_j, y_1, \ldots, y_r\end{pmatrix} \quad \text{for every } \xi \in \Xi$$

(*where* $i = 1, \ldots, r$, $j = 1, \ldots, l$) *are linearly independent solutions of the equations* $\xi A = 0$ *and* $Ax = 0$, *respectively.*

If x_0 *is orthogonal to* $\zeta_1, \ldots, \zeta_r$, *then the element* x *such that*

$$\xi x = \frac{1}{\delta}\bar{D}_{r+1}\begin{pmatrix}\xi, \eta_1, \ldots, \eta_r\\ Ux_0, y_1, \ldots, y_r\end{pmatrix} \quad \text{for every } \xi \in \Xi$$

is the only solution of the equation $Ax = x_0$ *orthogonal to* $\eta_1, \ldots, \eta_r, \omega_1, \ldots, \omega_l$, *where* $\omega_1, \ldots, \omega_l$ *are linearly independent solutions of the equation* $\xi U = 0$.

Analogously, if ξ_0 *is orthogonal to* $z_1, \ldots, z_{r+l}$, *then the element* ξ *such that*

$$\xi x = \frac{1}{\delta}\bar{D}_{r+1}\begin{pmatrix}\xi_0 U, \eta_1, \ldots, \eta_r\\ x, y_1, \ldots, y_r\end{pmatrix} \quad \text{for every } x \in X$$

is the only solution of the equation $\xi A = \xi_0$ *orthogonal to* $y_1, \ldots, y_r$.

(xi) *Let* $A \in \mathfrak{A}$ *be a generalized Fredholm operator such that* $r(A) = 0$, *and* $d(A) = l \geqslant 0$. *If* $\{D_n\}$ *is a determinant system for* A, *then*

$$D_n\begin{pmatrix}\xi_1, \ldots, \xi_{n+l}\\ x_1, \ldots, x_n\end{pmatrix} = \sum_p \operatorname{sgn} p D_0(\xi_{p_{n+1}}, \ldots, \xi_{p_{n+l}}) \det(\xi_{p_i} B x_j) \quad \text{for } n = 0, 1, \ldots$$

where $B \in \mathfrak{A}$ *is the quasi-inverse of* A *and* $\sum_p$ *is extended over all permutations* $p = (p_1, \ldots, p_{n+1})$ *of the integers* $1, \ldots, n+l$ *such that* $p_1 < \ldots < p_n$, $p_{n+1} < \ldots < p_{n+l}$.

4. Effective analytic formulae for determinant systems

Let two complex Banach spaces Ξ, X be conjugate (in the previously given sense) and let ξx be a bilinear functional such that

$$\|x\| = \sup_{\|\xi\| \leqslant 1} |\xi x|, \qquad \|\xi\| = \sup_{\|x\| \leqslant 1} |\xi x| \text{ for every } \xi \in \Xi, \quad x \in X.$$

Let us denote by $\mathfrak{M}$ the class of all bounded functionals F on $\mathfrak{A}$ such that operators (bilinear functionals) T_F, defined by the formula $\xi T_F x = F(x \cdot \xi)$ belong to A. If $F \in \mathfrak{M}$ is fixed, then we shall write T instead of T_F. Following Leżański [2] we shall also write $F_{\xi x}(\xi A x)$ instead of $F(A)$ for $A \in \mathfrak{A}$.

(xii) *Let $S \in \mathfrak{A}$ be a generalized Fredholm operator of the order zero, let $U \in A$ be a quasi-inverse of S, and let $s_1, \ldots, s_{d(S)}$ denote linearly independent solutions of $Sx = 0$. For any $F \in \mathfrak{M}$, let*

$$D_n(F) = \sum_{m=0}^{\infty} \frac{1}{m!} D_{n,m}(F),$$

where

$$D_{n,m}(F)\begin{pmatrix}\xi_1, \ldots, \xi_{n+d(S)} \\ x_1, \ldots, x_n\end{pmatrix} =$$

$$= F_{\eta_1 y_1} \ldots F_{\eta_m y_m} \Theta_{nm}\begin{pmatrix}\eta_1, \ldots, \eta_m, \xi_1, \ldots, \xi_{n+d(S)} \\ y_1, \ldots, y_m, x_1, \ldots, x_n\end{pmatrix} \quad \text{for } n = 0, 1, \ldots$$

The sequence $D_0(F)$, $D_1(F)$, ... is a determinant system for $S+T_F$ and does not depend on U. Moreover,

$$D_{0,m}(F) = \begin{vmatrix} \sigma_1, & m-1, & 0, & 0, \ldots, & 0 \\ \sigma_2, & \sigma_1, & m-2, & 0, \ldots, & 0 \\ \sigma_3, & \sigma_2, & \sigma_1, & m-3, \ldots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{m-1}, & \sigma_{m-2}, & \ldots, & \sigma_1, & 1 \\ \sigma_m, & \sigma_{m-1}, & \ldots, & \sigma_2, & \sigma_1 \end{vmatrix},$$

$$D_{n,m}(F) = \begin{vmatrix} T_n^0, & n, 0, \ldots, 0 \\ T_n^1 & \\ T_n^2 & \\ \vdots & D_{0,m}(F) \\ T_n^m & \end{vmatrix} \quad (n = 1, 2, \ldots)$$

where $\sigma_m = F((UT_F)^{m-1} U)$ for $m = 1, 2, \ldots$, and T_n^m is the $2n+d(S)$— linear functional

$$T_n^m\begin{pmatrix}\xi_1, \ldots, \xi_{n+d(S)} \\ x_1, \ldots, x_n\end{pmatrix} = \sum \det(a_{k,l}) = \sum \det(b_{k,l}),$$

where for $k, l = 1, \ldots, n+d(s)$

$$a_{k,l} = \begin{cases} \xi_k (UT_F)^{i_k} U x_l & \text{for } l \leq n \\ \xi_k (UT_F)^{i_k} s_l & \text{for } l > n \end{cases}$$

$$b_{k,l} = \begin{cases} \xi_k (UT_F)^{i_l} U x_l & \text{for } l \leq n \\ \xi_k (UT_F)^{i_l} s_l & \text{for } l > n \end{cases}$$

and summation Σ is extended over all sequences of non negative integers $i_1, \ldots, i_{n+d(S)}$ such that $i_1 + \ldots + i_{n+d(S)} = m$.

(xiii) *If $D_n(F)$ is a determinant system for $S+T$, then the sequences $\overline{D}_n(F)$ and $\underline{D}_n(F)$ defined as follows:*

$$\overline{D}_n(F)\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = D_n(F)\begin{pmatrix}\xi_1, \ldots, \xi_n, \omega_1, \ldots, \omega_{d(S)} \\ Sx_1, \ldots, \ldots \ldots Sx_n\end{pmatrix} \qquad (n = 0, 1, \ldots)$$

$$\underline{D}_n(F)\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = D_n(F)\begin{pmatrix}\xi_1 S, \ldots, \xi_n S, \omega_1, \ldots, \omega_{d(S)} \\ x_1, \ldots, \ldots \ldots x_n\end{pmatrix} \qquad (n = 0, 1, \ldots)$$

are determinant systems for $I+UT_F$ and $I+T_F U$, respectively, where $\omega_1, \ldots, \omega_{d(S)}$ are linearly independent solutions of the equation $\xi U = 0$ and such that $\omega s_j = \delta_{i,j}$ for $i, j = 1, \ldots, d(S)$.

Proofs of these theorems will be published in Studia Mathematica.

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MATHEMATICS

On Classes Λ_M^α and λ_M^α of 2π-Periodic Functions

by

R. TABERSKI

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1. Let M and N be the convex functions complementary in Young sense ([2], pp. 16—22). We shall denote by L_M^* the Orlicz space of 2π-periodic (measurable) functions f with norm $\|f\|_M = \sup_v \int_0^{2\pi} |f(x)|\, v(x)\, dx$, where the supremum is taken over all non-negative 2π-periodic (measurable) functions v such that $\int_0^{2\pi} N(v(x))\, dx \leqslant \leqslant 1$ ([2], pp. 83—9). The symbol L_∞^* will be used for the space of all 2π-periodic continuous functions f with norm $\|f\|_\infty = \max_{0\leqslant x\leqslant 2\pi} |f(x)|$.

Let $f \in L_M^*$ and let $f_h(x) = f(x+h) - f(x)$. We say that f belongs to the class $\Lambda_M^\alpha\ [\lambda_M^\alpha]$ $(0 < \alpha \leqslant 1)$ when

$$\|f_h\|_M = O(h^\alpha)\ [o(h^\alpha)] \text{ as } h \to 0+.$$

The class of functions $f \in L_\infty^*$ for which

$$\|f_h\|_\infty = O(h^\alpha)\ [o(h^\alpha)] \text{ as } h \to 0+$$

is denoted by $\Lambda_\infty^\alpha\ [\lambda_\infty^\alpha]$ $(0 < \alpha \leqslant 1)$. Obviously, $\Lambda_M^\beta \subset \lambda_M^\alpha \subset \Lambda_M^\alpha$, $\Lambda_\infty^\alpha \subset \Lambda_M^\alpha$ if $0 < \alpha \leqslant \beta \leqslant 1$.

By $\langle \Lambda_M^\alpha, \|\ \|_M^\alpha \rangle$ $(0 < \alpha \leqslant 1)$ we shall understand the linear space of functions $f \in \Lambda_M^\alpha$ with norm

$$\|f\|_M^\alpha = \sup_{0<h\leqslant 1} \frac{1}{h^\alpha} \|f_h\|_M.$$

In this space, two elements f and g are said to be identical if there is a constant c such that $f(x) = g(x) + c$, for almost every x; we define, as usual, the addition and multiplication of elements by numbers neglecting a constant term and the sets of measure zero. An arbitrary constant is the zero-element of this space. Analogous meanings have the symbols: $\langle \lambda_M^\alpha, \|\ \|_M^\alpha \rangle$, $\langle \Lambda_\infty^\alpha, \|\ \|_\infty^\alpha \rangle$. As it is easy to verify, λ_M^α is closed in $\langle \Lambda_M^\alpha, \|\ \|_M^\alpha \rangle$ and this remark remains true if M is replaced by ∞ ([3], § 1.0).

2. We shall present some properties of the classes Λ_M^α and λ_M^α $(0 < \alpha \leqslant 1)$. Arguing similarly as in [4] (p. 180, ex. 9), and applying the example 16 of [4] (p. 181), we obtain the following theorem.

2.1. *A necessary and sufficient condition for the function f to be in Λ_M^1 is that for almost every $x \in (-\infty, \infty)$*

$$f(x) = \int_0^x g(t)\,dt + c, \tag{1}$$

where $g \in L_M^*$, $\int_0^{2\pi} g(t)\,dt = 0$, and c is a constant.

The next theorem completes the above formulated result.

2.2. *If $f \in \Lambda_M^1$, $g \in L_M^*$, and if for almost every x the relation* (1) *holds, then*

$$\|f\|_M^1 = \|g\|_M. \tag{2}$$

Proof. Let v be non-negative, 2π-periodic, and such that $\int_0^{2\pi} N(v(x))\,dx \leqslant 1$. Then

$$\int_0^{2\pi} |f_h(x)|\, v(x)\,dx \leqslant \int_0^{2\pi} \left[\int_x^{x+h} |g(t)|\,dt\right] v(x)\,dx = \int_0^h \left[\int_0^{2\pi} |g(s)|\, v(s-t)\,ds\right] dt.$$

Hence

$$\frac{1}{h}\|f_h\|_M \leqslant \|g\|_M \text{ for } h > 0. \tag{3}$$

By Fatou's lemma

$$\varliminf_{h\to 0+} \int_0^{2\pi} \left|\frac{1}{h} f_h(x)\right| v(x)\,dx \geqslant \int_0^{2\pi} |g(x)|\, v(x)\,dx,$$

whence

$$\varliminf_{h\to 0+} \frac{1}{h}\|f_h\|_M \geqslant \|g\|_M. \tag{4}$$

From (3) and (4), the relation (2) follows.

2.3. *The space $\langle \Lambda_M^1, \|\ \|_M^1 \rangle$ is complete. This space is separable if and only if $M(u)$ satisfies the Δ_2-condition for large u* ([2], p. 35).

The last statement is an immediate consequence of the preceding theorems (see also [2], pp. 87—8, 103—4).

2.4. *If $M(u)$ satisfies the Δ_2-condition for large u, the spaces $\langle \lambda_M^\alpha, \|\ \|_M^\alpha \rangle$ $(0 < \alpha \leqslant 1)$ are separable.*

This theorem is an immediate consequence of 2.3 and the following lemma

2.5. *The set Λ_M^1 is derse in $\langle \lambda_M^\alpha, \|\ \|_M^\alpha \rangle$ if $0 < \alpha < 1$.*

Proof. Let $f \in \lambda_M^\alpha$. Consider the Stieklov's functions

$$f_n(x) = n \int_x^{x+1/n} f(t)\,dt \qquad (n = 1, 2, \ldots).$$

It is easy to observe that $f_n \in \Lambda_M^1$. Therefore, $f_n - f \in \lambda_M^\alpha$. Arguing similarly as in § 1.3 of [3], we obtain $\|f_n - f\|_M^\alpha \to 0$.

2.6. *If $f \in \Lambda_M^\alpha$ $(0 < \alpha \leqslant 1)$, then*

(5) $$\lim_{\varepsilon\to 0+} \|f\|_M^{\alpha-\varepsilon} = \|f\|_M^\alpha.$$

If $f \in \Lambda_M^\alpha (0 < \alpha < \beta \leqslant 1)$, then

(6) $$\lim_{\varepsilon\to 0+} \|f\|_M^{\alpha+\varepsilon} = \|f\|_M^\alpha.$$

Proof. The relation (5) [(6)] can be obtained applying the theorem 0.1 [0.2] of [3] to the function of h

$$\xi_h^\sigma(f) = \begin{cases} 0 & \text{for } h = 0, \\ h^{-\sigma} \sup\limits_v \int\limits_0^{2\pi} |f(x+h) - f(x)| v(x) dx & \text{for } 0 < h \leqslant 1, \end{cases}$$

where the supremum is taken over all non-negative and 2π-periodic functions v such that $\int\limits_0^{2\pi} N(v(x))\, dx \leqslant 1$ and $0 < \sigma \leqslant \alpha$ $[\alpha \leqslant \sigma < \beta]$.

3. Now, a theorem concerning the absolute convergence of Fourier series $S[f]$ will be given.

Denote by M^{-1} and N^{-1} the functions inverse to M and N, respectively.

3.1. *If $f \in \Lambda_M^1$ and if*

(7) $$\sum_{\nu=1}^{\infty} [N^{-1}(2^\nu)]^{-1/2} < \infty,$$

then $S[f]$ converges absolutely.

Proof. For almost every $x \in (-\infty, \infty)$, there holds the relation (1) with some $g \in L_M^*$ and some constant c. Without loss of generality of the proof, we may assume that for every x the representation (1) is true.

Let $\varrho_n = (a_n^2 + b_n^2)^{1/2}$, where a_n and b_n are Fourier coefficients of the function f. As is known,

(8) $$\sum_{n=1}^{\infty} \varrho_n^2 \sin^2 \frac{n\pi}{2^{\nu+1}} = \frac{1}{2^{\nu+3}\pi} \int\limits_0^{2\pi} \sum_{k=1}^{2^\nu+1} \left[f\left(x + \frac{k\pi}{2^\nu}\right) - f\left(x + \frac{(k-1)\pi}{2^\nu}\right)\right]^2 dx$$

for $\nu = 1, 2, \ldots$ ([4], pp. 241—2). The integrand of the right-hand side of (8) does not exceed

(9) $$\sum_{k=1}^{2^{\nu+1}} \left| f\left(x + \frac{k\pi}{2^\nu}\right) - f\left(x + \frac{(k-1)\pi}{2^\nu}\right)\right| \left| \int\limits_{\alpha_k}^{\beta_k} g(t)\, dt \right|,$$

where $\alpha_k = x + 2^{-\nu}(k-1)\pi$, $\beta_k = x + 2^{-\nu} k\pi$. By Hölder's inequality

(10) $$\left| \int\limits_{\alpha_k}^{\beta_k} g(t)\, dt \right| \leqslant \|g\|_M \cdot \frac{\pi}{2^\nu} M^{-1}\left(\frac{2^\nu}{\pi}\right).$$

In view of (10), (9) and (8),

$$\sum_{n=2^{\nu}-1}^{2^{\nu}-1} \varrho_n^2 \sin^2 \frac{n\pi}{2^{\nu+1}} \leqslant \frac{\pi}{2^{2(\nu+1)}} M^{-1}\left(\frac{2^{\nu}}{\pi}\right)\|g\|_M V \qquad (\nu = 1, 2, \ldots),$$

where V is the variation of f in the interval $\langle 0, 2\pi\rangle$. Hence,

$$\sum_{n=1}^{\infty} \varrho_n < \frac{1}{2}(\pi \|g\|_M V)^{1/2} \sum_{\nu=1}^{\infty} [2^{-\nu} M^{-1}(2^{\nu})]^{1/2}. \tag{11}$$

Since

$$2^{\nu} < M^{-1}(2^{\nu}) N^{-1}(2^{\nu}) \leqslant 2^{\nu+1} \qquad (\nu = 1, 2, \ldots), \tag{12}$$

the convergence of the series (7) implies the convergence of the series on the right-hand side of (11), and this completes the proof.

Remark. The assumption (7) is satisfied if $N(u)$ satisfies the Δ_2-condition for large u, because in this case $N(u)$ does not increase more rapidly than some power ([2], pp. 37—8). However, there exist other functions N for which the inequality (7) holds, e.g., this property have functions $N(u)$ equal to

$$e^{u^{\alpha}}\ (0 < \alpha < 1/2) \quad \text{and} \quad u^{\ln^{\beta} u} \qquad (\beta > 0)$$

for large u ([2], pp. 28, 49—50).

The last theorem completes the well-known results given in [4], pp. 241—2, (3.6), (3.8), (3.9).

Analogously, we can obtain the following statement for the series $H[f]$ of Haar coefficients a_n ([1]).

Denote by $L_M^*\langle o, 1\rangle$ the Orlicz space of measurable functions φ defined in the interval $\langle 0, 1\rangle$, with the norm

$$\|f\|_M = \sup_{v} \int_0^1 |\varphi(x)| v(x)\,dx \left(\int_0^1 N(v(x)\,dx \leqslant 1)\right).$$

3.2. *Let*

$$f(x) = \int_0^x g(t)\,dt + c$$

for all $x \in \langle 0, 1\rangle$, $g \in L_M^*\langle 0, 1\rangle$, $c = const$. *Suppose that the condition* (7) *is satisfied. Then* $H[f]$ *converges absolutely.*

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MATHEMATICS

Some Further Properties of φ-Functions

by

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1. Denote by S the class of continuous positive functions defined for $u > 0$, and define the following functions (which may assume also the value ∞):

$$\underline{h}_\varphi(\lambda) = \varliminf_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)}, \qquad \overline{h}_\varphi(\lambda) = \varlimsup_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)} \qquad \text{for } \lambda > 0.$$

A function $\varphi \in S$ will be called a quasi φ-function (briefly: $q\varphi$-function). if there exist the limits

$$(*) \quad s_\varphi = \lim_{\lambda\to 0+} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg \lambda}, \qquad (**)\ \sigma_\varphi = \lim_{\lambda\to 0+} \frac{\lg \overline{h}_\varphi(\lambda)}{-\lg \lambda},$$

finite or infinite. A function φ, continuous and nondecreasing for $u \geqslant 0$, vanishing at zero only and tending to ∞ as $u \to \infty$ is a $q\varphi$-function, and $\sigma_\varphi \geqslant s_\varphi \geqslant 0$ (cf. [5]). Such $q\varphi$-functions are called φ-functions according to the terminology of [4].

Nonincreasing functions of the class S are also $q\varphi$-functions, and $s_\varphi \leqslant \sigma_\varphi \leqslant 0$. To denote $q\varphi$-functions, we shall use Greek letters $\varphi, \psi, \chi, \varrho, \ldots$ In some cases the same symbols are to denote $\varphi \in S$.

Generalizing the definition from [4], functions $\varphi, \psi \in S$ will be said equivalent for large u (l-equivalent), in symbols $\varphi \overset{l}{\sim} \psi$, if

$$a\varphi(k_1 u) \leqslant \psi(u) \leqslant b\varphi(k_2 u) \quad \text{for } u \geqslant u_0,$$

a, b, k_1, k_2 being positive constants. It is easily seen that $\overset{l}{\sim}$ is an equivalence relation.

We shall give now some simple properties of the l-equivalence and of the indices s_φ, σ_φ.

1.1. If $\varphi \overset{l}{\sim} \psi$, where φ is a $q\varphi$-function, $\psi \in S$, then ψ is a $q\varphi$-function, and $s_\varphi = s_\psi$, $\sigma_\varphi = \sigma_\psi$.

1.2. (a) Let $\varphi_r(u) = (\varphi(u))^r$, $r \neq 0$; then $s_{\varphi r} = r s_\varphi$, $\sigma_{\varphi r} = r\sigma_\varphi$, and if $\varphi \overset{l}{\sim} \varphi_1$, then $\varphi_r \overset{l}{\sim} \varphi_{1r}$.

(b) If $\varphi(u) = u^r \psi(u)$, then $s_\psi = r + s_r$, $\sigma_\varphi = r + \sigma_\psi$ (the existence of indices on one side of these equations implies existence of the indices on the other side).

1.3. If $s_\varphi = \sigma_\varphi = r \neq 0$, then such a $q\varphi$-function will be called *quasiregularly increasing*; if $s_\varphi = \sigma_\varphi = 0$, *quasislowly varying* φ-functions of a regular increase in the sense of Karamata [2], i.e. $q\varphi$-functions satisfying the condition

(1.3.1) $$\varphi(u)/\varphi(\lambda u) \to g(\lambda) \text{ for } \lambda > 0,$$

where $g(\lambda)$ is finite and $\neq 0$ for every λ, $g(\lambda)$ not identically equal to 1, are quasiregularly increasing.

If $\lambda > 0$, from (1.3.1) it follows $g(\lambda) = \lambda^{-r}$, $r \neq 0$, for $g(\lambda_1 \lambda_2) = g(\lambda_1) g(\lambda_2)$ and $g(\lambda)$ is obviously of the first class of Baire. Hence, $s_\varphi = \sigma_\varphi = r$.

Assuming $g(\lambda) = 1$ for $\lambda > 0$, we obtain $q\varphi$-functions slowly varying in the sense of Karamata; then $s_\varphi = \sigma_\varphi = 0$. **1**.2 implies immediately

1.4. φ is quasiregularly increasing if and only if $\varphi(u) = u^r \psi(u)$, $r \neq 0$, where ψ is quasislowly varying.

1.5. If φ is a convex (concave) φ-function, then $\varphi \overset{l}{\sim} \varphi_1$, where φ_1 is a convex (concave) φ-function possessing a continuous, strictly increasing (decreasing) derivative for $u \geqslant 0$.

Let φ be a convex φ-function and let $\varphi(u) u^{-1} \to \infty$ as $u \to \infty$. Then $\varphi(u) u^{-1}$ is increasing for sufficiently large u and replacing φ by an equivalent function we may assume $\varphi(u) u^{-1}$ to be increasing for $u > 0$. Let $\varphi_1(u) = \int_0^u \varphi(t) t^{-1} dt$; since $\varphi(\frac{1}{2} u) \leqslant \varphi_1(u) \leqslant \varphi(u)$ for $u \geqslant 0$, φ_1 possesses the required properties. If $\varphi(u) u^{-1} \to d$, $d < \infty$, then $\varphi(u) \overset{l}{\sim} du$ and the function $\varphi_1(u) = du - d \lg(1+u)$ is l-equivalent to φ and satisfies the required conditions. If φ is concave, then it is strictly increasing. Since φ^{-1} is a convex φ-function, $\varphi^{-1} \overset{l}{\sim} (\varphi^{-1})_1$, where $(\varphi^{-1})_1$ denotes the function defined for φ^{-1} in the same way as φ_1 was defined for φ. Obviously, $((\varphi^{-1})_1)^{-1}$ possesses the required properties and by **1**.61, $\varphi \overset{l}{\sim} ((\varphi^{-1})_1)^{-1}$.

1.6. If $\varphi \overset{l}{\sim} \varphi_1$, $\varrho(u) = \int_0^u \varphi(t)\, dt$, $\varrho_1(u) = \int_0^u \varphi_1(t)\, dt$, $\varrho(u) \to \infty$, $\varrho_1(u) \to \infty$ as $u \to \infty$ for two $q\varphi$-functions φ, φ_1, then $\varrho \overset{l}{\sim} \varrho_1$.

The inequality

(**1**.6.1) $$a\varphi(k_1 u) \leqslant \varphi_1(u) \leqslant b\varphi(k_2 u) \text{ for } u \geqslant u_0$$

is satisfied for some positive constants a, b, k_1, k_2; hence,

$$ak_1^{-1} \int_{u_0 k_1}^{uk_1} \varphi(t)\, dt \leqslant \int_{u_0}^{u} \varphi_1(t)\, dt \leqslant bk_2^{-1} \int_{u_0 k_2}^{uk_2} \varphi(t)\, dt \qquad \text{for } u \geqslant u_0$$

and

$$\frac{1}{2}\, ak_1^{-1} \int_0^{uk_1} \varphi(t)\, dt \leqslant \int_0^{u} \varphi_1(t)\, dt \leqslant 2\, bk_2^{-1} \int_0^{uk_2} \varphi(t)\, dt$$

for $u \geqslant \bar{u}_0 \geqslant u_0$, where $\bar{u}_0$ is sufficiently large, i.e.

$$a'\varrho(k_1 u) \leqslant \varrho_1(u) \leqslant b'\varrho(k_2 u).$$

1.61. If $\varphi \overset{l}{\sim} \varphi_1$ and φ, φ_1 are strictly increasing φ-functions, then $\varphi^{-1} \overset{l}{\sim} \varphi^{-1}$.

φ, φ_1 satisfy the inequality (**1**.6.1); if $\varphi_1(u) = v$, $u = \varphi_1^{-1}(v)$, then the inequalities $k_1 u \leqslant \varphi^{-1}(a^{-1} v)$, $\varphi^{-1}(b^{-1} v) \leqslant k_2 u$, i. e. $k_2^{-1} \varphi^{-1}(b^{-1} v) \leqslant \varphi_1^{-1}(v) \leqslant \leqslant k_1^{-1} \varphi^{-1}(a^{-1} v)$ hold for $v \geqslant v_0 = \varphi_1(u_0)$.

1.7. If $\psi(u) = \int_0^u \varphi(t)\,dt$ is finite for $u > 0$, $\psi(u) \to \infty$, where φ is a $q\varphi$-function, then

(1.7.1) $$1 + s_\varphi \leqslant s_\psi, \qquad \sigma_\psi \leqslant 1 + \sigma_\varphi.$$

L'Hopital's rule yields $\bar{h}_\psi(\lambda) \leqslant \lambda^{-1} \bar{h}_\varphi(\lambda)$, $h_\psi(\lambda) \geqslant \lambda^{-1} \underline{h}_\varphi(\lambda)$, and it suffices to apply the definition of indices s and σ.

Remark. If φ is quasiregularly increasing, i.e. if $s_\varphi = s = r$, ψ is also quasiregularly increasing and $\leqslant$ in (1.7.1) may be replaced by $=$.

1.71. (a) If φ is a φ-function, ψ has the same meaning as in **1**.7, then

(**1**.71.1) $$1 + s_\varphi = s_\psi, \qquad 1 + \sigma_\varphi = \sigma_\varphi.$$

(b) If φ is a convex φ-function having a continuous derivative for $u \geqslant 0$, then

(**1**.71.2) $$s_\varphi = 1 + s_{\varphi'}, \qquad \sigma_\varphi = 1 + \sigma_{\varphi'}.$$

For a nondecreasing φ, the inequality $u\varphi(\frac{1}{2}u)/2 \leqslant \psi(u) \leqslant u\varphi(u)$ is satisfied for $u \geqslant 0$, whence $\psi \overset{l}{\sim} u\varphi$. Now, it is sufficient to apply **1**.1, and **1**.2 (b). The part (b) is a trivial consequence of (a).

Since, according to **1**.7 and [5], **2**.3 (b), $s'_\varphi > 0$ implies φ to be l-equivalent to a convex φ-function, owing to **1**.7 we obtain:

Formula (**1**.71.1) holds if φ' is a continuous at 0 $q\varphi$-function such that $s_{\varphi'} > 0$ or $s_{\varphi'} = \sigma_{\varphi'} = 0$ (in particular, if φ' is slowly varying).

1.8. Let φ be a strictly increasing φ-function; then $s_\varphi = 1/\sigma_{\varphi-1}$.

Let $\infty > s_\varphi > 0$ and $0 < s < s_\varphi$. By [5], **2**.3 (b), $\varphi \overset{l}{\sim} \chi_s$, $\chi_s = \psi(u^s)$, where ψ is a convex function. According to **1**.5 we may assume ψ to be strictly increasing. If $\chi_s(u) = v$, then $\psi^{-1}(v) = (\chi_s^{-1}(v))^s$, whence $\sigma_{\psi-1} = s\sigma_{\chi_s^{-1}}$, by **1**.2 (a). Since ψ^{-1} is a concave function, we have $\sigma_{\psi-1} \leqslant 1$ and by **1**.61, $\sigma_{\varphi-1} = \sigma_{\chi_s^{-1}}$, whence $\sigma_{\varphi-1} \leqslant \leqslant 1/s$. Thus $\sigma_{\varphi-1} \leqslant 1/s_\varphi$. If $s_\varphi = \infty$, the inequality $\sigma_{\varphi-1} \leqslant 1/s$ is satisfied for an arbitrary $s > 0$; hence $\sigma_{\varphi-1} = 0$. Let $0 < \sigma_\varphi < \infty$; assuming $\sigma_\varphi < \sigma$, we have $\varphi \overset{l}{\sim} \chi_\sigma$, $\chi_\sigma = \psi(u^\sigma)$, where ψ is concave. Arguing as above we state $s_{\varphi-1} \geqslant 1/\sigma$, whence $s_{\varphi-1} \geqslant 1/\sigma_\varphi$. If $\sigma_\varphi = 0$, σ may be taken arbitrarily small; consequently, $s_{\varphi-1} = \infty$. Applying the above proved inequality to φ^{-1}, we obtain $s_\varphi \geqslant 1/\sigma_{\varphi-1}$.

2. The following conditions play a role when investigating properties of $q\varphi$-functions:

$$(\infty_s)\ \lim_{u\to\infty} \varphi(u)\,u^{-s} = \infty, \qquad (\infty_\sigma^0)\ \lim_{u\to\infty} \varphi(u)\,u^{-\sigma} = 0,$$

$$(0_s)\ \lim_{u\to 0+} \varphi(u)\,u^{-s} = 0.$$

Denote $s_\varphi^* = \sup s$, where the sup is taken over exponents s such that (∞_s) holds, $\sigma_\varphi^* = \inf \sigma$, where σ are exponents satisfying (∞_σ^0).

2.1. The following inequalities hold for any φ-function: $s_\varphi \leqslant s_\varphi^* \leqslant \sigma_\varphi^* \leqslant \sigma_\varphi$.

Let $s_\varphi > 0$, $s < s' < s_\varphi$. By [5], **2**.3 (a), $\varphi \overset{l}{\sim} \chi_{s'}$, $\chi_{s'} = \psi(u^{s'})$, where ψ is a convex φ-function. Hence $\varphi(u) \geqslant a\psi(k_1^{s'} u^{s'})$ for $u \geqslant u_0$, whence $\varphi(u) u^{-s} = \varphi(u) u^{-s'} \cdot u^{s'-s} \geqslant a\psi(k_1^{s'} u^{s'}) u^{-s'} \cdot u^{s'-s} \geqslant a\psi(k_1^{s'} u_0) u_0^{-s'} u^{s'-s}$; thus (∞_s) is satisfied and $s_\varphi \leqslant s_\varphi^*$. Analogously we show $\sigma_\varphi^* \leqslant \sigma_\varphi$.

2.2. In this section we always assume φ to be a φ-function satisfying the conditions (0_1), (∞_1). Then a complementary function φ^* may be defined as follows:

$$\varphi^*(v) = \sup_{u \geqslant 0} (uv - \varphi(u)).$$

It is easily proved that φ^* is a φ-function satisfying (0_1), (∞_1) and that to every $v \geqslant 0$ there exists a u_v such that $\varphi^*(v) = u_v v - \varphi(u_v)$, [1], [7].

In the following we shall prove some theorems on complementary functions.

2.3. (a) *If* $\varphi_1(u) = a\varphi(bu)$, $a, b > 0$, *then* $\varphi_1^*(u) = a\varphi^*(u/ab)$.

(b) *If* $\varphi(u) \geqslant \varphi_1(u)$ *for* $u \geqslant u_0$, *then* $\varphi_1^*(u) \geqslant \varphi^*(u)$ *for* $u \geqslant u_0^*$.

(c) *If* $\varphi \overset{l}{\sim} \varphi_1$, *then* $\varphi^* \overset{l}{\sim} \varphi_1^*$.

To prove (a) note that $uv - a\varphi(bu) = a\left(bu\dfrac{v}{ab} - \varphi(bu)\right)$; hence $\varphi_1^*(v) = \sup\limits_{u \geqslant 0}(uv - a\varphi(bu)) = a \sup\limits_{u' \geqslant 0}\left(u'\dfrac{v}{ab} - \varphi(u')\right) = a\varphi^*(v/ab)$. As regards the proof of (b), cf. [6]. (a) and (b) imply (c) immediately.

2.4. The following formulae are satisfied for any convex φ-function:

$$(\mathrm{o})\, \frac{1}{s_{\varphi^*}} + \frac{1}{\sigma_\varphi}, \qquad (\mathrm{oo})\, \frac{1}{\sigma_{\varphi^*}} + \frac{1}{s_\varphi} = 1.$$

These inequalities are valid also in the limit cases, when the indices assume values 1, ∞, by usual conventions as regards the indeterminate expressions under consideration.

By **1**.5, taking into account **2**.3 (c) and the fact that s_φ and σ_φ are invariants of the relation $\overset{l}{\sim}$, we may assume φ to possess a derivative strictly increasing to ∞. Since $\varphi^*(u) = \int\limits_0^u (\varphi')^{-1}(t)\, dt$ is finite for convex φ-functions, we obtain the required formulae applying **1**.71 and **1**.8 successively.

The theorem may be proved also by applying [5], **1**.41 and **2**.3 (a) directly.

Formulae **2**.4 (o), (oo) are satisfied always, if $s_\varphi > 1$, for this condition implies $\varphi \overset{l}{\sim} \psi$, ψ is convex. Let $s_\varphi \leqslant 1$; the function $(\varphi^*)^* = \bar{\varphi}$ is called associated with the function φ (φ itself need not be convex). Obviously, $\bar{\varphi}$ is a convex φ-function satisfying (0_1), (∞_1); moreover [7], [5],

$$\bar{\varphi}(u) \leqslant \varphi(u) \text{ for } u \geqslant 0.$$

2.5. Inequalities $s_{\bar{\varphi}} \geqslant s_\varphi$, $\sigma_{\bar{\varphi}} \leqslant \sigma_\varphi$ are satisfied.

The first inequality is trivial; in fact, for $s_\varphi > 1$, $\varphi \overset{l}{\sim} \bar{\varphi}$, and always $s_{\bar{\varphi}} \geqslant 1$, $\bar{\varphi}$ being convex. To prove the second inequality suppose $\sigma_\varphi < \infty$ and note that $\sigma_\varphi = \inf \lg d_\alpha / \lg \alpha$ for an arbitrary φ-function, where inf is taken over all constants d_α, $\alpha > 1$, satisfying the inequality $\varphi(\alpha u) \leqslant d_\alpha \varphi(u)$ for $u \geqslant u(\alpha)$. Let $\varphi_1(u) =$

$= d_a^{-1}\varphi(au)$, where d_α, $a > 1$. If $\varphi_1(u) \leqslant \varphi(u)$ for $u \geqslant u(a)$, then by **2**.3 (b), $\varphi_1^*(u) = d_a^{-1}\varphi^*\left(\frac{d_\alpha}{a}u\right) \geqslant \varphi^*(u)$ for $u \geqslant u^*(a)$, and for $(\varphi^*)^* = \varphi$ and $\left(d_a^{-1}\varphi^*\left(\frac{d_\alpha}{a}u\right)\right)^* = d_a^{-1}\bar{\varphi}(au)$ there holds $d_a^{-1}\bar{\varphi}(au) \leqslant \bar{\varphi}(u)$ for $u \geqslant u_\alpha$, i.e. $\sigma_\varphi \leqslant$ $\leqslant \lg d_\alpha/\lg a$, $\sigma_{\bar{\varphi}} \leqslant \sigma_\varphi$.

2.6. The following inequality holds for an arbitrary φ-function (satisfying (0_1), (∞_1)):

$$\frac{1}{s_{\varphi^*}} + \frac{1}{\sigma_\varphi} \leqslant 1.$$

By the definition of φ and by 2.4, there holds $\frac{1}{s_{\varphi^*}} + \frac{1}{\sigma_{\bar{\varphi}}} = 1$ and it is sufficient to apply **2**.5.

It follows from **2**.4 and from [4] that the following properties are equivalent for convex φ [3]:

$$(\alpha) \qquad \varphi(2u) \leqslant d\varphi(u) \qquad \text{for } u \geqslant u_0,$$

$$(\beta) \qquad \varphi^*(au) \geqslant c_\alpha\varphi^*(u) \qquad \text{for } u \geqslant u^*(a), \quad \text{where } c_\alpha > a > 1.$$

If the convexity of φ is not assumed, (α) implies (β). Another trivial consequence of **2**.4 is: if φ is a convex pseudoregularly increasing φ-function, then φ^* has the same property.

3. Let a φ-function φ satisfy the conditions (0_1), (∞_1). Denote

$$(3.0.1) \qquad g(v) = \int_0^\infty e^{-\varphi(t)}e^{tv}\,dt \qquad \text{for } v \geqslant 0.$$

The function $\chi(v) = \lg g(v)/g(0)$ is a strictly increasing, convex φ-function satisfying the condition (∞_1) and $\varphi^* \overset{l}{\sim} \chi$. (cf. [8], where a theorem on analytic representation of a convex φ-function by means of a sum of a series of exponential functions is proved.).

Assuming t to be sufficiently large, we have $e^{-\varphi(t)}e^{tv} < e^{-vt-t}$; hence $g(v) < \infty$ for $v \geqslant 0$. Convexity of $\chi(v)$ is verified in a usual manner, e.g. applying Schwarz's inequality to $g(v)$. Let $0 < \lambda < 1$; the following inequality is satisfied for $v > -\lg\lambda$:

$$(*) \qquad g(v+\lg\lambda) = \int_0^\infty e^{-\varphi(t)+tv}e^{t\lg\lambda}\,dt \leqslant e^{\varphi^*(v)}\frac{1}{-\lg\lambda}.$$

Given $v \geqslant v_0$, choose u_v so that $\varphi^*(v) = u_v v - \varphi(u_v)$. If $v \geqslant v_0$, we have $u_v \geqslant 1$ and there holds the inequality

$$(**) \qquad g(v) \geqslant \int_{u_v-1}^{u_v} e^{-\varphi(t)}e^{tv}\,dt \geqslant e^{u_v v-\varphi(u_v)}e^{-v} = e^{\varphi^*(v)}e^{-v}.$$

If $v \geqslant v_1$, where v_1 is sufficiently large, we have $2v + \lg\lambda > v$ and by (*) and (**),

$$-v+\varphi^*(v) \leqslant \lg g(v) \leqslant \varphi^*(2v)+\lg(-\lg\lambda).$$

Taking into account that φ^* fulfills the condition (∞_1), the last inequality yields $\chi \overset{l}{\sim} \varphi^*$.

3.1. Let φ be an arbitrary φ-function. Suppose $0 < s_\varphi^- \leqslant 1$, $0 < s < s_\varphi$, then φ is l-equivalent to $\psi(u^s)$, where ψ is a convex φ-function. The function $\varphi_1(u) = \varphi(u^{1/s})$ is l-equivalent to ψ and satisfies (∞_1), for φ satisfies (∞_s), by 2. Moreover, one may suppose that φ_1 satisfies (0_1), replacing φ_1 by an l-equivalent function. Let $g_1(v)$ denote the integral (3.0.1), where φ is replaced by φ_1^*. By 3, $\bar{\varphi}_1 \overset{l}{\sim} \lg g_1(u)/\bar{g}_1(0) = \chi_1(u)$, where $\bar{\varphi}_1 = (\varphi_1^*)^*$. Since $\varphi_1 \overset{l}{\sim} \psi$, by 2.3 (c) $\bar{\varphi}_1 \overset{l}{\sim} \varphi_1$.

Since $\varphi(u) = \varphi_1(u^s)$, $\bar{g}_1$ is an integral function of the variable v, hence $\lg \bar{g}_1(u^s)/\bar{g}_1(0)$ is l-equivalent to $\varphi(u)$ and it is a locally analytic function for $u > 0$. If $\alpha, \beta \geqslant 0$, $\alpha^s+\beta^s = 1$, then taking into account that χ_1 increases monotonically, we obtain $\chi_1((\alpha v_1 + \beta v_2)^s) \leqslant \chi_1(\alpha^s v_1^s + \beta^s v_2^s) \leqslant \alpha^s \chi_1(v_1^s) + \beta^s \chi_1(v_2^s)$. Hence:

Let φ be an arbitrary φ-function satisfying (0_1) and let $s_\varphi > 0$, $s < s_\varphi$ when $s_\varphi \leqslant 1$, $s = 1$ when $s_\varphi > 1$ or $s_\varphi = 1$ and φ is equivalent to a convex φ-function satisfying (∞_1). Let $\varrho_s(v) = \sup_{u \geqslant 0}(uv - \varphi(u^{1/s}))$. By these assumptions

(a) $\varphi \overset{l}{\sim} \chi_\varphi$, where

$$\chi_\varphi(v) = \lg \Big(\int_0^\infty e^{-\varrho_s(t)} e^{tv^s}\, dt\Big) \Big/ \int_0^\infty e^{-\varrho_s(t)}\, dt;$$

(b) χ_φ is an s-convex function, i.e. $\chi_\varphi(\alpha v_1+\beta v_2) \leqslant \alpha^s \chi_\varphi(v_1)+\beta^s \chi_\varphi(v_2)$ for $\alpha, \beta \geqslant 0$, $\alpha^s+\beta^s = 1$, and χ_φ is locally analytic for $v > 0$.

Let us yet note that if $s_\varphi = 0$, then $u\varphi \overset{l}{\sim} \psi$, where $\psi(u) = \int_0^u \varphi(t)\, dt$, whence ψ is convex, and by the previous theorem we obtain also in this case existence of locally analytic functions l-equivalent to φ, defined by integrals of the above type with a factor u^{-1}.

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MATHEMATICS

On Direct Decompositions of Complete Direct Sums of Groups of Rank 1

by

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Introduction. All groups considered in this paper are Abelian. If $\{H_t\}_{t\in T}$ (where T is a set of indices) is a family of groups, then $\sum\limits_{t\in T}^{*} H_t$ denotes the complete direct sum of H_t for $t \in T$, i.e., the group of all functions x defined on T such that $x(t) \in H_t$, for every $t \in T$. $\sum\limits_{t\in T} H_t$ denotes the discrete direct sum of H_t, i.e., the subgroup of $\sum\limits_{t\in T}^{*} H_t$ composed of all such functions x that are equal to zero for all but a finite number of indices $t \in T$.

Let $\{H_t\}_{t\in T}$ be a family of torsion free groups of rank 1. It is known [4], [3], that

(i) Every direct summand of $\sum\limits_{t\in T} H_t$ is isomorphic to $\sum\limits_{t\in T_1} H_t$, where T_1 is a subset of T.

In this paper we impose some additional conditions on the family $\{H_t\}_{t\in T}$ and then we prove that every direct summand of $\sum\limits_{t\in T}^{*} H_t$ is isomorphic to $\sum\limits_{t\in T_1}^{*} H_t$, for some $T_1 \subset T$.

1. Let G and H be two groups. Then there exists a canonical homomorphism $g: G \to \mathrm{Hom}\,(\mathrm{Hom}\,(G, H), H)$ defined by $(g(a))(h) = h(a)$, for every $h \in \mathrm{Hom}\,(G, H)$ and $a \in G$.

We say that G is H-dual if the homomorphism g is an isomorphism.

Let H be a torsion free group of rank 1 and let $\mathfrak{a} = (a_1, a_2, ...)$ be the type of H. Let $\mathfrak{a}^\circ = (b_1, b_2, ...)$, where $a_i = b_i$, if $a_i = \infty$, and $b_i = 0$, if $a_i \neq \infty$. Then H° denotes a torsion free group of rank 1 and of the type $\mathfrak{a}^\circ$.

It is easy to see that H° is isomorphic to a subgroup of H, $\mathrm{Hom}\,(H, H) = H^\circ$ and $\mathrm{Hom}\,(H^\circ, H) = H$.

LEMMA 1. *Every torsion free group H of rank 1 is H-dual.*

Proof. $g: H \to \mathrm{Hom}\,(\mathrm{Hom}\,(H, H), H)$ is a monomorphism, since we have $(g(a))(i) = i(a) = a$, for the identity map i of H. Now, let $h \in \mathrm{Hom}\,(\mathrm{Hom}\,(H, H), H)$ and suppose that $h(i) = a$. Then $h(i) = i(a)$ and so $h(i) = (g(a))(i)$. But

$\operatorname{Hom}(H, H) = H^\circ$ and H is torsion free. Hence, if two homomorphisms of $\operatorname{Hom}(H, H)$ into H coincide on one element that is different from zero, then they coincide on $\operatorname{Hom}(H, H)$. Therefore $h = g(a)$ and thus g is an isomorphism.

LEMMA 2. *Every direct summand of an H-dual group G is H-dual.*

Proof. Let $G = G_1 + G_2$ be a direct decomposition of G. Then $\operatorname{Hom}(G, H) = \operatorname{Hom}(G_1, H) + \operatorname{Hom}(G_2, H)$ is a direct decomposition of $\operatorname{Hom}(G, H)$, where $\operatorname{Hom}(G_i, H)$, for $i = 1, 2$, is considered as the subgroup of $\operatorname{Hom}(G, H)$ composed of all such homomorphisms that are trivial on G_{3-i}. It is easy to see that the canonical homomorphism $g: G_1 \to \operatorname{Hom}(\operatorname{Hom}(G_1, H), H)$ is a monomorphism. Hence it suffices to show that for every homomorphism f_1 of $\operatorname{Hom}(G_1, H)$ into H there exists an element $a_1 \in G_1$ such that $f_1(h) = h(a_1)$, for every $h \in \operatorname{Hom}(G_1, H)$. Let $f_1 \in \operatorname{Hom}(\operatorname{Hom}(G_1, H)$. Then f_1 can be extended to a homomorphism f of $\operatorname{Hom}(G, H)$ into H. But G is H-dual. Therefore there exists an element $a \in G$ such that $f(h) = h(a)$ for every $h \in \operatorname{Hom}(G, H)$. The element a can be represented in the form $a_1 + a_2$, where $a_i \in G_i$, for $i = 1, 2$. If $h \in \operatorname{Hom}(G_1, H)$ then $f_1(h) = f(h) = h(a_1 + a_2) = h(a_1) + h(a_2) = h(a_1)$. Thus $f_1(h) = h(a_1)$ and the Lemma is proved.

Let $\{H_t\}_{t \in T}$ be a family of groups. Then for every subset $T_1 \subset T$, $\sum^*_{t \in T_1} H_t$ $(\sum_{t \in T_1} H_t)$ will be considered as the subgroup of $\sum^*_{t \in T} H_t$ $(\sum_{t \in T} H_t)$ composed of all functions $x \in \sum^*_{t \in T} H_t$ $(x \in \sum_{t \in T} H_t)$ that are equal to zero for all $t \in T - T_1$. In particular H_α will be considered as a subgroup $\sum_{t \in T} H_t$, for any $\alpha \in T$.

The following easy and well known property is stated here for reference:

(ii) Let $\{H_t\}_{t \in T}$ be a family of groups and let H be a group. Consider the mapping $\varphi: \operatorname{Hom}(\sum_{t \in T} H_t, H) \to \sum^*_{t \in T} \operatorname{Hom}(H_t, H)$ defined by $(\varphi(f))(t) = f|H_t$ for every $f \in \operatorname{Hom}(\sum_{t \in T} H_t, H)$ and $t \in T$. Then φ is an isomorphism.

Let $\{Z_t\}_{t \in T}$ be a countable family of infinite cyclic groups. A torsion free group H is called slender if every homomorphism of $\sum^*_{t \in T} Z_t$ into H sends all but a finite number of components Z_t into zero.

The following properties of slender groups are known:

(iii) (see [2], Th. 47.2, p. 170). Let $\{H_t\}_{t \in T}$ be a family of torsion free groups and suppose that the cardinality of T is of measure zero. Let H be a slender group and let ψ be the mapping of $\operatorname{Hom}(\sum^*_{t \in T} H_t, H)$ into $\sum^*_{t \in T} \operatorname{Hom}(H_t, H)$ defined by $(\psi(f))(t) = f|H_t$. Then ψ is an isomorphism of $\operatorname{Hom}(\sum^*_{t \in T} H_t, H)$ onto $\sum_{t \in T} \operatorname{Hom}(H_t, H)$

(iv) (see [2], Th. 47.4, p. 172). The direct sum of slender groups is again slender.

(v) (see [5]) Every countable reduced torsion free group is slender.

LEMMA 3. *Let $\{H_t\}_{t \in T}$ be a family of torsion free groups and suppose that the cardinality of T is of measure zero. Let H be a slender group and let H_t be H-dual, for every $t \in T$. Then $\sum^*_{t \in T} H_t$ and $\sum_{t \in T} H_t$ are H-dual.*

The above lemma follows easily from (ii) and (iii).

2. THEOREM 1. *Let $\{H_t\}_{t\in T}$ be a family of torsion free groups of rank* 1 *satisfying the following conditions*

a) *the cardinality of T is of measure zero,*

b) *H_α is not a divisible group for any $\alpha \in T$,*

c) *for any $\alpha, \beta \in T$, either H_α is isomorphic to H_β or H_α^0 is not isomorphic to any subgroup of H_β.*

Then any direct summand of $\sum\limits_{t\in T}^* H_t$ is isomorphic to $\sum\limits_{t\in T_1}^* H_t$, where T_1 is a subset of T.

Proof. Let T_0 be a subset of T satisfying the following conditions:

1) there exists a mapping τ from T onto T_0 such that H_t is isomorphic to $H_{\tau(t)}$ for every $t \in T$;

2) if $\alpha \neq \beta$; $\alpha, \beta \in T_0$, then H_α is not isomorphic to H_β.

Then for any two different elements $\alpha, \beta \in T_0$, H_α^0 is not isomorphic to any subgroup of H_β and $\mathrm{Hom}(H_t, H_\beta) = \mathrm{Hom}(H_t^0, H_\beta) = 0$ whenever $\tau(t) \neq \beta$. Therefore $\mathrm{Hom}(H_t, \sum\limits_{u\in T_0} H_u) = \mathrm{Hom}(H_t, H_{\tau(t)}) = H_t^0$ and $\mathrm{Hom}(H_t^0, \sum\limits_{u\in T_0} H_u) = \mathrm{Hom}(H_t^0, H_{\tau(t)}) = H_t$. Hence H_t is $\sum\limits_{u\in T_0} H_u$-dual. Now, it follows from Lemma 3 that $\sum\limits_{t\in T}^* H_t$ is also $\sum\limits_{u\in T_0} H_u$-dual, since (iv) and (v) imply that $\sum\limits_{u\in T_0} H_u$ is slender. Therefore, by Lemma 2, we obtain that any direct summand of $\sum\limits_{t\in T}^* H_t$ is $\sum\limits_{u\in T_0} H_u$-dual. Let G_1 be a direct summand of $\sum\limits_{t\in T}^* H_t$. Then $\mathrm{Hom}(G_1, \sum\limits_{u\in T_0} H_u)$ is a direct summand of $\mathrm{Hom}(\sum\limits_{t\in T}^* H_t, \sum\limits_{u\in T_0} H_u) = \sum\limits_{t\in T} \mathrm{Hom}(H_t, \sum\limits_{u\in T_0} H_u) = \sum\limits_{t\in T} H_t^0$. Hence, (i) gives that $\mathrm{Hom}(G_1, \sum\limits_{u\in T_0} H_u)$ is isomorphic to $\sum\limits_{t\in T_1} H_t^0$, where T_1 is a subset of T. Therefore $\mathrm{Hom}(\mathrm{Hom}(G_1, \sum\limits_{u\in T_0} H_u), \sum\limits_{u\in T_0} H_u) = \mathrm{Hom}(\sum\limits_{t\in T_1} H_t^0, \sum\limits_{u\in T_0} H_u) = \sum\limits_{t\in T_1}^* \mathrm{Hom}(H_t^0, \sum\limits_{u\in T_0} H_u) = \sum\limits_{t\in T_1}^* \mathrm{Hom}(H_t^0, H_{\tau(t)}) = \sum\limits_{t\in T_1}^* H_t$. But G_1 is isomorphic to $\mathrm{Hom}(\mathrm{Hom}(G_1, \sum\limits_{u\in T_0} H_u), \sum\limits_{u\in T_0} H_u)$. Thus, G_1 is isomorphic to $\sum\limits_{t\in T_1}^* H_t$.

As a particular case of the above theorem we obtain the following corollary

COROLLARY. *Let $\{H_t\}_{t\in T}$ be a family of reduced torsion free groups of rank* 1 *and of the same type $\mathfrak{a}$. Suppose that the cardinality of T is of measure zero. Then every direct summand of $\sum\limits_{t\in T}^* H_t$ is isomorphic to $\sum\limits_{t\in T_1}^* H_t$, where T_1 is a subset of T.*

Remark. Let $\{H_t\}_{t\in T}$ and $\{H_t\}_{t\in T'}$ be two families of torsion free groups of rank 1. Let $r(\mathfrak{a})$ and $r'(\mathfrak{a})$ denote the number of groups of type $\mathfrak{a}$ in the family $\{H_t\}_{t\in T}$ and $\{H_t\}_{t\in T'}$, respectively. If $\sum\limits_{t\in T} H_t$ is isomorphic to $\sum\limits_{t\in T'} H_t$ then $r(\mathfrak{a}) = r'(\mathfrak{a})$, for every type $\mathfrak{a}$ (see [1], Cor. 2.9 or [2], Th. 46.1).

Combining the above result and (i) one can easily obtain that

(vi) if $\{H_t\}_{t\in T}$ is a family of torsion free groups of rank 1 then, for every direct summand of $\sum_{t\in T} H_t$, there exist an automorphism g of $\sum_{t\in T} H_t$ and a subset $T_1 \subset T$ such that $g(G_1) = \sum_{t\in T} H_t$.

Now, using (vi) and applying the technique used in the proof of Theorem 1 one can obtain the following result

THEOREM 2. *Let $\{H_t\}_{t\in T}$ be a family of torsion free groups of rank 1 satisfying conditions a)—c) of Theorem 1. If G_1 is a direct summand of $\sum^*_{t\in T} H_t$, then there exists an automorphism g of $\sum^*_{t\in T} H_t$ and a subset $T_1 \subset T$ such that*

$$g(G_1) = \sum_{t\in T_1}^* H_t.$$

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MATHEMATICS

On Modified Landau Polynomials

by

J. RADECKI

Presented by W. ORLICZ on May 3, 1961

Let $h_n > 0$ and $h_n \to \infty$ as $n \to \infty$. Modified Landau polynomials are defined as follows

$$P_n[f(t); x] = \frac{\int\limits_0^1 f(h_n t)\left[1-\left(t-\frac{x}{h_n}\right)^2\right]^n dt}{2\int\limits_0^1 (1-t^2)^n\, dt}.$$

THEOREM 1. *We have $P_n(1; x) \to 1$ as $n \to \infty$ at a point $x > 0$ if and only if*

$$\frac{h_n}{\sqrt{n}} \to 0.$$

THEOREM 2. *If* α) $\frac{h_n}{\sqrt{n}} \to 0$ *as* $n \to \infty$,

β) *the sequence δ_n satisfies the conditions*

$$\delta_n > 0, \qquad \delta_n \to 0, \qquad \delta_n \frac{\sqrt{n}}{h_n} \to \infty,$$

γ) *the function $f(t)$ is measurable and bounded in every interval $[a, b]$ where $a \geqslant 0$,*

δ) *the function $f(t)$ is continuous at $x > 0$,*
then $P_n(f; x) \to f(x)$ as $n \to \infty$ if and only if

$$J_n(x) = \frac{\sqrt{n}}{h_n} \int\limits_{x+\delta_n}^{h_n} f(t)\left[1-\left(\frac{t-x}{h_n}\right)^2\right]^n dt \to 0.$$

THEOREM 3. *If the assumptions* α) *and* β) *of Theorem 2 are satisfied and if the function $f(t)$ is continuous in the interval $[0, \infty)$, then the polynomials $P_n(f; x)$ are convergent to $f(x)$ almost uniformly in $[0, \infty)$ if and only if the sequence $J_n(x)$ tends to zero almost uniformly in $[0, \infty)$.*

THEOREM 4. *If* α) $\frac{h_n^s}{n} \to 0$ *as* $n \to \infty$, *where* $s \geqslant 2$,

β) *there exist constants* M, m *such that* $|f(t)| \leqslant Me^{mt^s}$ *for every* $t \geqslant 0$,

γ) *the function* $f(t)$ *is measurable in every interval* $[a, b]$, *where* $a \geqslant 0$,

δ) *the function* $f(t)$ *is continuous at the point* $x > 0$,

then $P_n(f; x) \to f(x)$.

THEOREM 5. *If the assumptions* α) *and* β) *of Theorem* 4 *are satisfied and the function* $f(t)$ *is continuous in the interval* $[0, \infty)$, *then the polynomials* $P_n(f; x)$ *are convergent to* $f(x)$ *almost uniformly in* $[0, \infty)$.

THEOREM 6. *If the assumptions* α), β) *and* γ) *of Theorem* 4 *are satisfied and if the derivative* $f^{(2k)}(x)$ *exists at the point* $x > 0$, *then*

$$\lim_{n\to\infty}\left(\frac{\sqrt{n}}{h_n}\right)^{2k}\left\{P_n[f(t); x] - \sum_{r=0}^{2k-1}\frac{f^{(r)}(x)}{r!}P_n[(t-x)^r; x]\right\} = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}\frac{f^{(2k)}(x)}{(2k)!}.$$

The proofs of the above Theorems will appear in Studia Mathematica

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MATHEMATICS

The Complements of Bounded, Open, Connected Subsets of Euclidean Space

by

M. K. FORT, Jr.

Presented by K. KURATOWSKI on March 6, 1961

Let G_1 and G_2 be bounded, connected open sets in Euclidean n-space E^n such that G_1 and G_2 are homeomorphic. We show that $E^n - G_1$ and $E^n - G_2$ have the same number of components. (If infinite, cardinal numbers are the same). The question as to whether or not this was so was raised by K. Kuratowski and was communicated to the author by A. Granas. The theorem does not seem to follow from the standard results of algebraic topology.

LEMMA. *Let G be a bounded connected open set in Euclidean n-space, and let p and q be points in some component A of the boundary of G. Corresponding to each $\varepsilon > 0$ there is a $\delta > 0$ such that if $p' \in N(\delta, p) \cap G$ and $q' \in N(\delta, q) \cap G$, then there is an arc α having end points p' and q' such that $\alpha \subset N(\varepsilon, \text{boundary } G) \cap G$.*

Proof. Suppose $\varepsilon > 0$. We define K to be the set of all points x in G such that the distance from x to the boundary of G is not less than ε. The set K is compact, and hence there exists a set P such that: P is the union of a finite number of closed cubes, $P \subset G$, and K is contained in the interior of P. Choose a positive number δ less than the distance from P to the boundary of G. Now suppose that $p' \in N(\delta, p) \cap G$ and $q' \in N(\delta, q) \cap G$. Since G is connected, there is an arc β_1 in G having end points p' and q'.

If β_1 is disjoint from P, then we may let $\alpha = \beta_1$. Otherwise order β_1 from p' to q' and let s_1 be the first point of $\beta_1 \cap P$. Let P_1 be the component of P which contains s_1 and let t_1 be the last point of $\beta_1 \cap P_1$. The continuum A is disjoint from P_1, and hence is contained in some component R_1 of $E^n - P_1$. Both t_1 and s_1 are on the boundary of R_1, and by the Brouwer Property (See [1], p. 47), the boundary J_1 of R_1 is a connected subset of P_1. We may now replace the part of β between t_1 and s_1 by an arc in J_1 and obtain an arc β_2 from p to q such that β_2 is disjoint from the interior of P_1.

If β_2 is disjoint from $P - P_1$, we define $\alpha = \beta_2$. Otherwise, let t_2 be the first point of $\beta_2 \cap (P - P_1)$, let P_2 be the component of P which contains t_2, and let s_2 be the last point of $\beta_2 \cap P_2$. By the same argument used above, we may replace the part of β_2 between s_2 and t_2 by an arc in the boundary of P_2 and obtain an arc β_3 from p' to q' which is disjoint from the interior of $P_1 \cup P_2$.

Continuing in this manner, we obtain after a finite number of steps an arc β_k, where k is not more than the number of components of P, such that β_k is disjoint from the interior of P. We may let $\alpha = \beta_k$.

THEOREM. *Let G_1 and G_2 be bounded and connected open subsets of E^n. If G_1 is homeomorphic to G_2, then the set of components of $E_n - G_1$ has the same cardinal number as the set of components of $E^n - G_2$.*

Proof. Let h be a homeomorphism on G_1 onto G_2, and let C_i be the set of components of $E^n - G_i$ for $i = 1, 2$. We will define a one-to-one function φ having domain C_1 and range C_2. Let $x \in C_1$. There exists a point $p_x \in x \cap$ (boundary G_1) and a sequence $s_1, s_2, \ldots$ in G_1 which converges to p_x. We may assume that $s_1, s_2, \ldots$ is chosen such that $h(s_1), h(s_2), \ldots$ converges to some point q_x. The point q_x must be in the boundary of G_2, and we define $\varphi(x)$ to be the component of $E^n - G_2$ that contains q_x.

We wish to show that the definition of $\varphi(x)$ is independent of the particular point p_x chosen in x and is independent of the particular sequence $s_1, s_2, \ldots$ Thus, let p_x and p'_x be points in $x \cap$ (boundary G_1), and let $s_1, s_2, \ldots$ and $s'_1, s'_2, \ldots$ be sequences converging to p_x and p'_x respectively such that $h(s_1), h(s_2), \ldots$ and $h(s'_1), h(s'_2), \ldots$ converge to points q_x and q'_x, respectively. We must show that q_x and q'_x belong to the same component of $E^n - G_2$. By our Lemma, there exists a sequence $\alpha_1, \alpha_2, \ldots$ of arcs in G_1 such that the end points of α_i are s_i and s'_i and $\limsup_{i\to\infty} \alpha_i$ is contained in x. It follows that $\limsup_{i\to\infty} h[\alpha_i]$ is a connected subset of the boundary of G_2 and that this set contains both q_x and q'_x. Thus q_x and q'_x belong to the same component of $E^n - G_2$.

We next prove that the range of φ is all of C_2. Let $y \in C_2$. There exists a point $q^* \in y \cap$ (boundary G_2) and a sequence $t_1, t_2, \ldots$ in G_2 which converges to q^*. We may suppose $t_1, t_2, \ldots$ chosen so that $h^{-1}(t_1), h^{-1}(t_2), \ldots$ converges to some point p^* in the boundary of G_1. Let x be the component of $E^n - G_1$ that contains p^*. It follows from the preceding paragraph that we may let $p_x = p^*$ and $s_i = h^{-1}(t_i)$. Thus $q_x = q^*$ and $\varphi(x) = y$.

Finally, we must show that φ is one-to-one. Suppose $x, x' \in C_1$ and $\varphi(x) = \varphi(x')$. We will show that $x = x'$. Let us choose points $q \in \varphi(x) \cap$ (boundary G_2) and $q' \in \varphi(x') \cap$ (boundary G_2). There exist sequences $t_1, t_2, \ldots$ and $t'_1, t'_2, \ldots$, in G_2 which converge to q and q', respectively, and which are chosen so that $h^{-1}(t_1)$, $h^{-1}(t_2), \ldots$ and $h^{-1}(t'_1), h^{-1}(t'_2), \ldots$ converge respectively to points $p \in x \cap$ (boundary G_1) and $p' \in x' \cap$ (boundary G_1).

Now, by the Lemma, we may choose arcs $\beta_1, \beta_2, \ldots$ in G_2 such that the end points of β_1 are t_i and t_i' and $\limsup_{i\to\infty} \beta_i \subset \varphi(x)$.

It follows that $\limsup_{i\to\infty} h^{-1}(\beta_i)$ is a connected subset of boundary G_1 which contains both p and p'. This implies that $x \cap x' \neq \emptyset$, and hence $x = x'$.

COROLLARY 1. *If G_1 and G_2 are open, connected subsets of the n-sphere S^n and G_1 is homeomorphic to G_2, then $S^n - G_1$ and $S^n - G_2$ have the same number of components.*

Proof. Let H be a homeomorphism of G_1 onto G_2. Choose a point $p \in G_1$ and let $q = H(p)$. We let F_p be a homeomorphism on $S^n - \{p\}$ onto E^n and we let F_q be a homeomorphism on $S^n - \{q\}$ onto E^n. Let V be a spherical neighborhood of p such that $\bar{V} \subset G_1$ and let $W = H(V)$. Then $F_p(G_1 - \bar{V})$ and $F_q(G_2 - \bar{W})$ are homeomorphic and are bounded, open, connected subsets of E^n. It is clear that $E^n - F_p(G_1 - \bar{V})$ has "one more" component than $S^n - G_1$ and that $E^n - F_q(G_2 - \bar{W})$ has "one more" component than $S^n - G_2$. It follows from our theorem that $E^n - F_p(G_1 - \bar{V})$ and $E^n - F_q(G_2 - \bar{W})$ have the same number of components, hence $S^n - G_1$ and $S^n - G_2$ have the same number of components.

Remark. Let φ be the function defined in our Theorem which sets up a one-to-one correspondence between the components of $E^n - F_p(G_1 - \bar{V})$ and those of $E^n - F_q(G_2 - \bar{W})$. We define a one-to-one function Φ on the set of components of $S^n - G_1$ onto the set of components of $S^n - G_2$ by letting $\Phi = F_q^{-1} \varphi F_p$.

DEFINITION. If G is an open connected subset of S^n, we define $D(G)$ to be the upper semi-continuous decomposition space whose members are the points of G and the components of $S^n - G$. It is clear that G is a subspace of $D(G)$ as well as a subspace of S^n.

COROLLARY 2. *If G_1 and G_2 are open, connected subsets of S^n and H is a homeomorphism G_1 onto G_2, then H may be extended to a homeomorphism of $D(G_1)$ onto $D(G_2)$.*

Proof. We let Φ be defined as in our remark, and then let

$$H^*(x) = \begin{cases} H(x) \text{ if } x \in G_1 \\ \Phi(x) \text{ if } x \text{ is a component of } S^n - G_1. \end{cases}$$

It is not difficult to verify that H^* is a homeomorphism of $D(G_1)$ onto $D(G_2)$.

COROLLARY 3. *Let K_1 and K_2 be compact, zero dimensional sets which are imbedded in S^n, $n > 1$. Then K_1 and K_2 are homeomorphic and are imbedded in the same manner if and only if $S^n - K_1$ is homeomorphic to $S^n - K_2$.*

Proof. If K_1 and K_2 are homeomorphic and are imbedded in the same manner then there is a homeomorphism of S^n onto itself which maps K_1 onto K_2. The same homeomorphism obviously maps $S^n - K_1$, onto $S^n - K_2$.

Now suppose that H is a homeomorphism of $S^n - K_1$ onto $S^n - K_2$. We let $G_1 = S^n - K_1$ and $G_2 = S^n - K_2$. Then G_1 and G_2 are open, connected sets and we may apply Corollary 2 to obtain a homeomorphism H^* on $D(G_1)$ onto $D(G_2)$ which maps K_1 onto K_2. However, since K_1 and K_2 are compact and zero dimensional, $D(G_1)$ and $D(G_2)$ are both the same as S^n *).

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THEORETICAL PHYSICS

A Special Model of a Two-Group Approach in Neutron Transport Theory

by

R. ŻELAZNY and A. KUSZELL

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Introduction

The purpose of this paper is to discuss a special model of a two-group approach in the neutron transport theory, which is characterized by the relation $l_1 = l_2$, where l_i denotes a mean free path for the i-th energetical group of neutrons.

This model is very interesting since, as we have shown, it can be solved exactly. For illustration we have discussed two interesting examples: the solution of Milne's problem for a half space and the solution of a critical problem for a slab. These examples enable us to deduce some interesting remarks valuable for general considerations and possible practical purposes.

The considered model has been suggested by Davison's well known monograph [1], where chiefly the existence of such a model in connection with Milne's problem treated there by the classical Wiener—Hopf method has been discussed. The approach used by us is based on the Case method [2] and a paper of one of the authors [3], which allowed us to treat the non-classical slab problem.

Solution of the system of Boltzmann equations

In the case of $l_1 = l_2$ the system of two-group Boltzmann equations in plane geometry can be written in the form:

$$(1) \qquad \left(\mu \frac{\partial}{\partial x} + 1\right) \Psi_i(x, \mu) = \frac{c_{ik}}{2} \int_{-1}^{1} \Psi_k(x, \mu')\, d\mu',$$

where x is measured in the appropriate units and μ is a cosine of the angle between neutron velocity and the x axis. c_{ik} is a matrix describing a transfer of neutrons from the k-th to the i-th group. All its elements are obviously non-negative.

Let us find a matrix u_{kl} which has the following property

$$(2) \qquad u_{ik}\, c_{kl} = \lambda_{[i]}\, \delta_{ik}\, u_{kl}\,.$$

The index in brackets [ ] denotes that there is no summation with respect to this index. This matrix must fulfil the following equation:

(3) $$u_{ik}\left[c_{kl}-\lambda_{[i]}\,\delta_{kl}\right]=0.$$

The non-trivial solutions of Eq. (3) exist, when

(4) $$\lambda_{1/2}=\frac{c_s\pm\sqrt{c_s^2-4c}}{2}\underset{\mathrm{def}}{=}\frac{1}{2}\,c_{1/2},$$

where $c_s=c_{11}+c_{22}$, $c=\det c_{kl}$.

We can limit ourselves to the case when c_{12} and c_{21} do not vanish simultaneously, because in this situation the matrix c_{kl} is diagonal (there is no coupling between two-group equations). Without any loss of generality we can assume that $c_{12}\neq 0$ and in this case we can choose arbitrary constants in the definition of u_{kl} in such a way that

(5) $$u_{kl}=\begin{pmatrix}-c_{12}, & c_{11}-\lambda_1\\ -c_{12}, & c_{11}-\lambda_2\end{pmatrix}.$$

This matrix has a determinant equal to

(6) $$u=-c_{12}\,(\lambda_1-\lambda_2),$$

which does not vanish when $\lambda_1\neq\lambda_2$ or in other words when

$$c_s^2-4c=(c_{11}-c_{22})^2+4c_{12}\,c_{21}\neq 0$$

(this expression is non-negative, thus λ_{12} are always real quantities). The latter condition holds when $c_{11}\neq c_{22}$ or $c_{12}\cdot c_{21}$ does not vanish simultaneously. We shall assume that this condition is always satisfied. This enables us to conclude that the matrix u_{kl} exists.

Now we can use Case's standard ansatz:

(7) $$\Psi_i(x,\mu)=e^{-\frac{x}{\nu}}\,\Phi_i(\mu,\nu),$$

which inserted into Eq. (1) gives us the following set of equations for Φ_i:

(8) $$\left(1-\frac{\mu}{\nu}\right)\Phi_i(\mu,\nu)=c_{ik}\int_{-1}^{1}\Phi_k(\mu',\nu)\,d\mu'.$$

Let us multiply Eq. (8) by u_{li} and put under the integral sign the expression $\overset{-1}{u_{ks}}\cdot u_{sn}=\delta_{kn}$. We obtain

(9) $$\left(1-\frac{\mu}{\nu}\right)u_{li}\,\Phi_i(\mu,\nu)=u_{li}\,c_{ik}\cdot\overset{-1}{u_{ks}}\int_{-1}^{1}u_{sn}\,\Phi_n(\mu',\nu)\,d\mu'.$$

Using the notation

(10) $$\xi_l(\mu,\nu)=u_{li}\,\Phi_i(\mu,\nu),$$

Eq. (9) can be rewritten in the form

(11) $$\left(1-\frac{\mu}{\nu}\right)\xi_l(\mu,\nu)=\frac{1}{2}c_{[l]}\int_{-1}^{1}\xi_l(\mu',\nu)\,d\mu',$$

where the notation from Eqs. (2) and (4) has been used. It is easily seen that Eq. (11) has the form of a separated one-velocity Boltzmann equation [2] and its solutions can be written in the form

(12) $$\xi_l(\mu,\nu)=\left[\frac{c_l}{2}\frac{\nu}{\nu-\mu}+\lambda_{[l]}(\nu)\,\delta(\nu-\mu)\right]a_l(\nu),$$

(13) $$\lambda_l(\nu)=1-\frac{c_l}{2}\nu f(\nu),$$

(14) $$f(\nu)=\int_{-1}^{1}\frac{d\mu}{\nu-\mu},$$

where $\nu\in(-1,1)$, which corresponds to the well known fact that $\xi(\mu,\nu)$ is a continuous eigenfunction of Eq. (11). Besides, there is also a discrete set of eigenfunctions, corresponding to $2N$ discrete eigenvalues given by the roots of the equation

(15) $$\lambda_l(\pm\nu_k)=0$$

(it should be kept in mind that $\lambda_l(\nu)$ is an even function of ν). When $c_l\leqslant 0$, Eq. (15) has no solutions. Discrete eigenfunctions have the form $[c_l>0]$:

(16) $$\xi_k(\mu,\pm\nu_l)=\frac{c_l}{2}\frac{\nu_{[l]}}{\nu_{[l]}\pm\mu}a_{[l]}(\pm\nu_l)\,\delta_{kl}$$

The appropriate completeness and orthogonality theorems are given by Case [2].

On the basis of these eigenfunctions the general solution of Eqs. (1) can be constructed:

(17) $$\Psi_i(x,\mu)=\underset{\nu}{S}\,e^{-\frac{x}{\nu}}\Phi_i(\mu,\nu)=\underset{\nu}{S}\,u_{ik}\,e^{-\frac{x}{\nu}}\,\xi_k(\mu,\nu)\underset{\text{def}}{=}u_{ik}\,\zeta_k(x,\mu),$$

where $\underset{\nu}{S}$ denotes a summation with respect to all discrete eigenvalues and an integration in the interval $(-1,1)$ over the continuous spectrum of eigenvalues.

Milne's problem

Let us formulate the boundary conditions for Milne's problem. From the general solution (17) of the Boltzmann equations we can deduce the following, most general behaviour of $\Psi_i(x,\mu)$ at infinity:

(18) $$\Psi_i(x,\mu)\sim\sum_{k=1}^{N}u_{ik}^{-1}\,a_k\,\xi_k(\mu,-\nu_k)\,e^{\frac{x}{\nu_k}}$$

corresponding to different kinds of sources at infinity. These different sources are generated by different values of constants $a_k(-\nu_k)$. The boundary condition on a free surface $x=0$ must be also used:

(19) $$\Psi_i(0,\mu)=0\quad\text{for}\quad\mu>0.$$

Now we can rewrite these two conditions in the following way:

(20)
$$\begin{cases} \text{a)} \quad \zeta_k(x,\mu) \underset{x\to\infty}{\to} a_{[k]}(-\nu_k)\,\xi_k(\mu,-\nu_k)\,e^{\frac{x}{\nu_k}}, \\ \text{b)} \quad \zeta_k(0,\mu) = 0 \quad \text{for} \quad \mu > 0, \end{cases}$$

where the multiplication of Eqs. (18) and (19) by the matrix u_{kl} and the definition of ζ_k from Eq. (17) have been used.

Let us observe that the function $\zeta_k(x,\mu)$ is a general solution of the one-velocity Boltzmann equation with c_k as a mean number of secondaries. Conditions (20) formulate Milne's problem for this equation. It is well known that Milne's problem for the one velocity Boltzmann equation is solved completely by means of the classical Wiener—Hopf method (see for example Davison's monograph) or by means of the eigenfunction expansion method given in the quoted work of Case.

Thus Milne's problem in the two-group neutron transport theory is reduced to two independent one-velocity Milne's problems with appropriate c_k. This implies that the extrapolation lengths should be defined not for $\Psi_l(x,\mu)$ but for $\zeta_k(x,\mu)$ functions, since in the latter situation the extrapolation lengths will not depend on the strength of sources at infinity. Thus

(21)
$$Z_{0k} = \frac{1}{2}\nu_{[k]}\ln\frac{\nu_k+1}{\nu_k-1} - \frac{1}{\pi}\int_1^\infty \frac{ds}{s^2-\frac{1}{\nu_{[k]}^2}}\tan^{-1}\left\{\frac{c_k\pi}{2s-2c_k\ln\frac{s+1}{s-1}}\right\}.$$

Critical problem for a slab

Let us solve the system of two-group Boltzmann equations in the region $-d \leqslant x \leqslant d$ with the following boundary conditions

(22)
$$\begin{cases} \Psi_k(-d,\mu) = \quad \text{for} \quad \mu > 0 \\ \Psi_k(d,\mu) = 0 \quad \text{for} \quad \mu > 0 \end{cases}.$$

According to the discussion of the critical problem for a slab given by one of the authors [3] let us use the symmetry property of functions $\Psi_k(x,u)$:

(23)
$$\Psi_k(x,\mu) = \Psi_k(-x,-\mu).$$

Conditions (22) can be reduced now to one condition in the form

(24)
$$\Psi_k(d,\mu) = 0 \quad \text{for} \quad \mu < 0$$

with additional use of property (23), which can be formulated only in the point $x = 0$:

(25)
$$\Psi_k(0,\mu) = \Psi_k(0,-\mu).$$

Using the procedure identical with that used in the preceding paragraph we can rewrite these conditions by help of the $\zeta_k(x,\mu)$ functions:

(26)
$$\begin{cases} \zeta_k(d,\mu) = 0 \quad \text{for} \quad \mu < 0 \\ \zeta_k(0,\mu) = \zeta(0,-\mu). \end{cases}$$

As previously, function $\zeta_k(x, \mu)$ is a solution of the one-velocity Boltzmann equation with a c_k given by definition (4). Thus the critical problem for a slab in two-group approach has been reduced to two independent critical problems for one-velocity Boltzmann equations. Let us discuss the possible situations. Since always $c_1 > c_2$ we can distinguish the following cases:

$$\text{a)} \begin{cases} c_1 > 1 \\ c_2 > 1 \end{cases} \quad \text{b)} \begin{cases} c_1 > 1 \\ 0 < c_2 < 1 \end{cases} \quad \text{c)} \begin{cases} c_1 > 1 \\ c_2 < 0 \end{cases} \quad \text{d)} \begin{cases} c_1 < 1 \\ 0 < c_2 < 1 \end{cases} \quad \text{e)} \begin{cases} c_1 < 1 \\ c_2 < 0 \end{cases}.$$

In the case a) there are two critical values of d : d_1 which corresponds to a critical solution for ζ_1 and d_2, which corresponds to the critical solution for ζ_2. From the relation between c_k mentioned above one can establish that $d_1 < d_2$. Thus, only d_1 has a physical meaning, since for slabs of thickness $2d_2$ we shall have a non-stationary solution for ζ_1, increasing exponentially with time.

In case b) there is only one critical value: d_1, corresponding to the critical problem for ζ_1. The equation for ζ_2 has no stationary solution for any value of d, since $0 < c_2 < 1$.

Case c) is completely analogical to case b), though $c_2 < 0$. Simple consideration devoted to the non-stationary problem for one-velocity Boltzmann equation with $c_2 < 0$ convinces that there exist no stationary solutions for any d. This case differs from the previously discussed one by the fact that there exist no discrete solutions. This situation corresponds to the well known discussion in Davison's monograph devoted to the absence of diffusion solutions in the multigroup approach.

Cases d) and e) are characterized by the absence of any stationary solutions. To case e) one must add consideration used in case c), since $c_2 < 0$. Both cases correspond to subcritical situations.

Thus the problem of an evaluation of critical values of d has been reduced to the one-velocity critical problem for a slab solved already in [3].

Appropriate formulae can be found there. We must add only that $\zeta_2(x, \mu) \equiv 0$ in all discussed cases. This can be written in the form

$$\Psi_1(x, \mu) \propto \Psi_2(x, \mu) \tag{27}$$

which corresponds to the well known fact in the diffusion theory that for bare critical assemblies $\varrho_1(x) \propto \varrho_2(x)$, where $\varrho_k(x)$ is a neutron density (in diffusion approximation).

Conclusions

The model of the multigroup neutron transport theory discussed in the paper in the case $l_1 = l_2$, though of very limited applications, can be solved exactly, and this enables us to formulate very interesting remarks of a transport character.

It seems that the formulation of extrapolation length for special linear combinations of neutron groups, instead for the groups themselves can be adapted also to other situations and should be used in practical considerations, as for example for calculation of control rods.

Our opinion is that our consideration devoted to the critical problem for a slab can, at least partly, be also transposed to other situations. Thus the mechanism of criticality problems has become somewhat clearer.

A more general consideration connected with multigroup approach in neutron transport theory is in preparation.

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THEORETICAL PHYSICS

On Some Representation of Perturbation Expansion of Scattering Amplitude

by
W. GARCZYŃSKI

Presented by L. INFELD on April 20, 1961

1. Let G denote connected Feynman diagram with L internal lines, E — external lines and V — vertices. External lines as well as external vertices have the same labels. Therefore the number of first internal vertex is equal $E+1$.

Contribution of diagram G to the collision amplitude is as follows:

$$F(G) = \lim_{\epsilon \to +0} \int dq_1 \dots \int dq_L \cdot \prod_{r=1}^{V} \delta\,[e_l^r\, q_l + \theta(E-r)\, p_r] \prod_{l=1}^{L} (q_l^2 - m_l^2 + i\epsilon)^{-1}. \tag{1.1}$$

The product of δ-functions expresses the momentum conservation law in any vertex.

The coefficients e_l^r can acquire the values 1, -1 or 0, respectively, if the line with l-number is incoming in r vertex, outgoing or avoiding one [1]. We assume here that all masses $m_l > 0$. After integration over dependent internal momenta $q_1, \dots, q_{V-1}$ we obtain

$$F(G) = \delta\left(\sum_{\substack{k=1\\ \epsilon \to +0}}^{E} p_k\right) \cdot \int dq_V, \dots \int dq_L \cdot \prod_{l=1}^{L} (q_l^2 - m_l^2 + i\epsilon)^{-1}, \tag{1.2}$$

where

$$q_l = \sum_{r=V}^{L} P_l^r \cdot q_r + \sum_{s=1}^{V-1} Q_l^s \cdot p_s \qquad (l = 1, \dots, L), \tag{1.3}$$

$$P_l^r = \begin{cases} \widetilde{P}_l^r & l \leqslant V-1 \\ \delta_{rl} & l > V-1 \end{cases}; \qquad Q_l^s = \begin{cases} \widetilde{Q}_l^s & l \leqslant V-1 \\ 0 & l > V-1 \end{cases}. \tag{1.4}$$

Obviously, the $\widetilde{P}_l^r$, $\widetilde{Q}_l^s$ can be equal to 1, -1 or 0, [2].

Now let us use the α-representation for causal propagator [1]

$$(q_l^2 - m_l^2 + i\epsilon)^{-1} = \frac{1}{i} \int_0^{\infty} d\alpha_l \exp i\alpha_l (q_l^2 - m_l^2 + i\epsilon). \tag{1.5}$$

The contribution $F(G)$ becomes

(1.6) $$F(G)=\left(\frac{1}{i}\right)^{L}\cdot\delta(\Sigma p)\int dq_V\ldots\int dq_L\int_0^\infty da_1\ldots\int_0^\infty da_L\exp i\sum_{l=1}^{L}a_L(q_l^2-m_l^2+i\epsilon).$$

2. We consider the quadratic form $\varphi=\sum_{l=1}^{L}\xi_l(q_l^2-m_l^2-i\epsilon)$, where $\xi_1,\ldots,\xi_l$ are some parameters and q_l is defined by formula (1.3). Using (1.3) φ can be brought to the form:

(2.1) $$\varphi=\sum_{r,r'=V}^{L}A_{rr'}\,q_r\cdot q_{r'}+2\sum_{r=V}^{L}B_r\,q_r+C.$$

where

(2.2) $$A_{rr'}=\sum_{l=1}^{L}\xi_l P_l^r\cdot P_l^{r'},$$

(2.3) $$B_r=\sum_{l=1}^{L}\sum_{j=1}^{V-1}\xi_l P_l^r\,Q_l^j\,p_j,$$

(2.4) $$C=\sum_{l=1}^{L}\sum_{j,j'=1}^{V-1}\xi_l Q_l^j Q_l^{j'}\,p_j p_{j'}-\sum_{l=1}^{L}(m_l^2-i\epsilon)\xi_l.$$

Substitution $q=q'-A^{-1}B$ gives

(2.5) $$\varphi=(Aq',q')+C-(B,A^{-1}B).$$

This is always possible because A has an inverse.

LEMMA 1. *The determinant of algebraic equations system $Aa+B=0$ does not vanish when the parameters $\xi_l\neq 0$ $(l=V,\ldots,L)$.*

Proof. Let us assume that $\det A=0$. Hence, the rows of A are linearly dependent

(2.6) $$a_V A_{Vj}+\ldots+a_L A_{Lj}=0\qquad(j=V,\ldots,L).$$

From (3.2) we have

(2.7) $$\sum_{l=1}^{L}\xi_l P_l^j[a_V P_l^V+\ldots+a_L P_l^L]=0\qquad(j=V,\ldots,L).$$

The coefficients for independent ξ_l parameters, $\neq 0$, vanish

(2.8) $$P_l^j[a_V P_l^V+\ldots+a_L P_l^L]=0\qquad(j,l=V,\ldots,L).$$

If we put $j=l=V$, we obtain $a_V=0$. Analogously, for $j=l=V+k$ we obtain $a_{V+k}=0$. This contradicts the linear dependence of rows and the Lemma 1 is proved.

LEMMA 2. *The quadratic form* $\varphi_M = \sum_{r,r'=V}^{L} A_{rr'} k_r k_{r'}$, *where* k_r *are numbers, is positive definite, if* $\xi_l > 0$ $(l = 1, ..., L)$.

Proof. Let us consider the quadratic form

$$f = \sum_{l=1}^{L} \xi_l k_l^2 \geqslant 0, \tag{2.9}$$

$$k_l = \sum_{r=l}^{L} P_l^r k_r + \sum_{s=1}^{V-1} Q_l^s p_s \qquad (l = 1, ..., L). \tag{2.10}$$

The substitution $k = k' - A^{-1} B$ gives

$$f = (Ak', k') + \widetilde{C} - (B, A^{-1} B). \tag{2.11}$$

The orthogonal transformation $k'' = L^{-1} k'$ is a cardinal axis transformation

$$f = \sum_{l=V}^{L} \lambda_l (k_l'')^2 + \widetilde{C} - (B, A^{-1} B), \tag{2.12}$$

where λ_l are characteristic values of matrix A. From Lemma 1 we obtain $\lambda_l \neq 0$. From inequality $f \geqslant 0$ we obtain

$$\widetilde{C} - (B, A^{-1} B) \geqslant 0 \text{ and } \lambda_l > 0. \tag{2.13}$$

From the Silvester criterion it follows that general minors of A are positive, when all ξ_l are positive.

LEMMA 3. *The relation* $C - (B, A^{-1} B) = \frac{D}{M}$ *is valid, if* $M = \det A$. D *is determinant of the matrix*

$$D = \begin{vmatrix} A_{VV}, ..., A_{VL}, B_V \\ \cdots\cdots \\ \cdots\cdots \\ A_{LV}, ..., A_{LL}, B_L \\ B_V, ..., B_L, \quad C \end{vmatrix}. \tag{2.14}$$

Proof. $D = \sum_{r=V}^{L} B_r [D_{r,L+1}] + C \cdot M$, where $[D_{r,L+1}]$ denote cofactor of $D_{r,L+1}$ element.

$$[D_{r,L+1}] = (-1)^{r+L} \cdot \sum_{s=V}^{L} (-1)^{r+L} \cdot B_s [A_{rs}] = -\sum_{s=V}^{L} B_s [A_{rs}], \tag{2.15}$$

$$D = C \cdot M - \sum_{r,s=V}^{L} B_r B_s [A_{rs}] = M\left(C - \sum_{r,s=V}^{L} B_r B_s A_{sr}^{-1}\right). \tag{2.16}$$

$D = M[C - (B, A^{-1} B)]$ and Lemma 3 is proved.

3. If the diagrams index satisfies inequality $\omega(G) = 4C_L - 2L < 0$ (where C_L denotes the number of closed loops in diagram), the contribution $F(G)$ converges. In the case, when $\omega(G) \geqslant 0$, the $F(G)$ is infinite and it is necessary to introduce the regularization procedure. As a result of regularization we obtain [1]

$$(3.1)\qquad \operatorname{reg} F(G) = \left(\frac{1}{i}\right)^{L+C} \cdot \delta(\Sigma p) \int dq_V \ldots \int dq_L \int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_L \prod_{s=1}^{L} I(\alpha_s) \times$$
$$\times \exp i \sum_{l=1}^{L} \alpha_l (q_l^2 - m_l^2 + i\epsilon),$$

$$(3.2)\qquad I(\alpha_s) = 1 + \sum_{j=1}^{h} C_j^s \exp i\alpha_s (m_s^2 - M_j^2).$$

M_j are regularising masses, introduced by Pauli and Villars [1]. The coefficients C_j^s are defined by set of equations

$$(3.3)\qquad \left.\begin{array}{l} 1 + \sum_{j=1}^{h} C_j^s = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ m_s^{2(h-1)} + \sum_{j=1}^{h} C_j^s M_j^{2(h-1)} = 0 \end{array}\right| \quad (s = 1, \ldots, L).$$

From (3.3) it follows simply that $I(\alpha_s)$ has a zero-point in $\alpha_s = 0$, degree h.

The change of variables $q = Lq' - A^{-1}B$ diagonalize the quadratic form and integration over the internal momenta may be made with the help of Gauss integral [1].

$$(3.4)\qquad \int e^{i(aq^2+bq)}\, dq = \frac{\pi^2}{ia^2} e^{-\frac{ib^2}{4a}} \quad a > 0.$$

Finally we obtain

$$(3.5)\qquad \operatorname{reg} F(G) = \pi^{2C} \left(\frac{1}{i}\right)^{L+C} \cdot \delta(\Sigma p) \int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_L \frac{I(\alpha_1)\ldots I(\alpha_L)}{M^2(\alpha)} \exp i \frac{D(\alpha)}{M(\alpha)}.$$

The above form of $F(G)$ may be used for investigation of analytic properties of perturbation expansion of scattering amplitude. From (2.13) it follows clearly that the part of $\frac{D(\alpha)}{M(\alpha)}$ which does not include the masses is positive definite for the external momenta. It is easy to show that $M(\alpha)$, $D(\alpha)$ are homogeneous functions of α degrees C and $C+1$, respectively.

If $\omega(G) < 0$, the regularization is not necessary. By the simple exchange of variables

$$(3.6)\qquad \alpha_l = \lambda \beta_l, \qquad \sum_{l=1}^{L} \beta_l = 1, \qquad (l = 1, \ldots, L)$$

with Jacobian $\dfrac{\partial(\alpha_1, \dots, \alpha_L)}{\partial(\beta_1, \dots, \beta_{L-1}, \lambda)} = \lambda^{L-1}$ and after integration over λ we obtain the Chisholm formulae, [4].

$$F(G) = (i\pi^2)^C (L-2C-1)! \int_0^1 d\beta_1 \dots \int_0^1 d\beta_L \, \delta\left(1 - \sum_{l=1}^{L} \beta_l\right) \frac{M(\beta)^{L-2C-2}}{D(\beta)^{L-2C}} . \tag{3.7}$$

However, if $\omega(G) > 0$, the regularization is necessary. In this case we obtain

$$\operatorname{reg} F(G) = \frac{\pi^{2C} \cdot i^{C-2L}}{(2C-L)!} \int_0^1 d\beta_1 \dots \int_0^1 d\beta_L \, \frac{\delta\left(1 - \sum\limits_{l=1}^{L} \beta_l\right)}{M^2(\beta)} \sum_{j_1=0}^{h} \dots \sum_{j_L=0}^{h} \tag{3.8}$$

$$\times C^1_{j_1} \dots C^L_{j_L} \left[\sum_{s=1}^{L} \beta_s (m_{js}^2 - M_{js}^2) + \frac{D(\beta)}{M(\beta)}\right]^{\frac{\omega(G)}{2}} \cdot \ln \left| \sum_{s=1}^{L} \beta_s (m_{js}^2 - M_{js}^2) + \frac{D(\beta)}{M(\beta)} \right|$$

$$h = \frac{\omega(G)}{2} + 1, \qquad m_0 = M_0 = 0, \qquad C_0^s = 1, \qquad (s = 1, \dots, L).$$

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Some Topological Properties of Feynman Diagrams

by

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1. Structure of any connected Feynman diagram can be described with the help of the following natural numbers:

L — number of internal lines,

V — number of vertices in diagram,

E — number of external lines.

An important characteristic of the diagram is the number of closed loop

$$C = L - V + 1. \tag{1,1}$$

The lines and vertices may be enumerated arbitrarily. However, it is convenient to assume that numbers of external lines and of corresponding vertices are the same. If for a certain vertex the number of incoming lines is greater than one, we indicate these external lines with the help of two labels.

For example, the lines incoming to vertex with number S are denoted as follows

$$S1, S2, ..., SE_s. \tag{1.2}$$

We see that E_s external lines are incoming to vertex with index S. The vertex to which the external line is incoming we call external, all remaining being called internal vertices.

We begin the enumeration of vertices from the external vertex. As a result, first internal vertex, if such exists, has a number $E+1$.

On the lines we choose an arbitrary direction by means of the arrows. All external lines are incoming.

Every internal line corresponds to the four-moment q and the mass $m > 0$. The momentum conservation law for any vertex is

$$\sum_{l=1}^{L} e_l^r q_l + \sum_{k=1}^{E} f_k^r p_k = 0 \quad (r = 1, ..., V), \tag{1,3}$$

where the coefficients e_l^r determine the direction of lines and are equal to 1, -1 or 0, if the line with number l is incoming to vertex r, outgoing, or avoiding it.

Coefficients f_k^r are equal to 1, or 0, because all external lines are incoming. p_k denote integer external momenta of k-line

$$p_k = p_{k1} + \ldots + p_{kE_k}. \tag{1,4}$$

In our case

$$f_k^r = \delta_{rk}\, \theta(E-r), \tag{1,5}$$

where

$$\theta(E-r) = \begin{cases} 1 & r \leqslant E \\ 0 & r > E. \end{cases} \tag{1,6}$$

The momenta conservation law becomes

$$\sum_{l=1}^{L} e_l^r q_l + \theta(E-r) p_r = 0 \quad (r = 1, \ldots, V). \tag{1,7}$$

We call the matrix of coefficients e_l^r — the matrix of diagram and denote it by $e(G)$. This matrix is investigated in topology [1].

$$e(G) = \| e_l^r \| \quad \begin{pmatrix} r = 1, \ldots, V \\ l = 1, \ldots, L \end{pmatrix}. \tag{1,8}$$

In every column there are only two terms not equal to zero, with oppositive signs. Hence, we can write

$$\sum_{r=1}^{V} e_l^r = 0 \quad (l = 1, \ldots, L), \tag{1,9}$$

From Eq. (1.7) we obtain

$$\sum_{l=1}^{L} \left(\sum_{r=1}^{V} e_l^r \right) q_l + \sum_{r=1}^{V} \theta(E-r) p_r = 0. \tag{1,10}$$

The first term vanishes and we obtain

$$\sum_{r=1}^{E} p_r = 0. \tag{1,11}$$

This is necessary for internal consistence of system (1.7) and expresses the external momenta conservation law.

2. We see, that the range of $e(G)$ matrix is equal to $V-1$ while our diagram is connected. Without loss of generality we can write

$$\det e_{V-1} = \begin{vmatrix} e_1^1, \ldots, & e_{V-1}^1 \\ \cdots & \cdots \\ \cdots & \cdots \\ e_1^{V-1}, \ldots, & e_{V-1}^{V-1} \end{vmatrix} \neq 0. \tag{2,1}$$

The solution of Eq. (1,7) is [5]

(2,2) $$q_k = \sum_{j=1}^{V-1} \widetilde{Q}^j_k p_j + \sum_{l=V}^{L} \widetilde{P}^l_k q_l \qquad (k = 1, ..., V-1),$$

where

(2,3) $$\widetilde{Q}^j_k = -\frac{[e^j_k]\,\theta(E-j)}{\det e_{V-1}} \qquad (j, k = 1, ..., V-1)$$

(2,4) $$\widetilde{P}^l_k = -\frac{\sum\limits_{j=1}^{V-1} [e^j_k]\, e^j_l}{\det e_{V-1}}$$

and $[e^j_k]$ denotes cofactor of e^j_k element.

Now we demonstrate some properties of diagram's matrix using the method of induction proposed by Bogoliubov and Parasiuk [2].

LEMMA 1. *The determinant of e_{V-1} can be equal either to 1 or —1.*

Proof. The Lemma is valid for a simple diagram. We assume that the Lemma is true for a diagram with V vertices and L internal lines. Now we put the new internal line with number $L+1$ between two given vertices. The matrix e_{V-1} in the course of operation is unchanged and the Lemma is valid in this case.

If we put in the diagram the new vertex with number 0 and the new line connected with the first vertex, we obtain

(2,5) $$e_V(G) = \left\| \begin{matrix} e^0_0, \ldots, & e^0_{V-1} \\ e^1_0, \ldots, & e^1_{V-1} \\ \cdots & \cdots \\ e^{V-1}_0, \ldots, & e^{V-1}_{V-1} \end{matrix} \right\| .$$

From this it follows immediately that,

(2,6) $$\det e_V = e^0_0 \det e_{V-1}, \qquad \text{where} \quad e^0_0 = \pm 1 .$$

By repeating these two operations we can construct every diagram. Therefore, Lemma 1 is valid for any connected diagram. In a similar manner we can demonstrate the following

LEMMA 2. *The cofactors of matrix e_{V-1} elements may equal* 1, — 1 *or* 0.

THEOREM. *Coefficients $\widetilde{Q}^j_k$ and $\widetilde{P}^l_k$ defined by formulas* (2,3) *and* (2,4) *may be equal to* 1, — 1 *or* 0.

Proof. The Theorem is valid for a simple diagram. We assume that the Theorem is true for a diagram with V vertices and L internal lines. Now we put the new internal line with number $L+1$ between any two vertices. Without loss of generality we can assume that the line connects vertex with number 1 and vertex with number V. In this case we obtain

(2,7) $$\widetilde{P}^l_k = -\frac{\sum\limits_{j=1}^{V-1} [e^j_k]\, e^j_l}{\det e_{V-1}} \qquad \begin{pmatrix} k = 1, \ldots, V-1 \\ l = V, \ldots, L+1 \end{pmatrix}.$$

In the whole course of this operation the matrix e_{V-1} is unchanged and therefore the coefficients $\widetilde{Q}_k^j$ and $\widetilde{P}_k^l$ for $(l = V, ..., L)$ are also unchanged.

Now we ought to consider the coefficients P_k^{L+1} $(k = 1, ..., V-1)$. We obtain

$$\widetilde{P}_k^{L+1} = -\frac{\sum_{j=1}^{V-1} [e_k^j]\, e_{L+1}^j}{\det e_{V-1}} = -\frac{[e_k^1]\, e_{L+1}^1}{\det e_{V-1}}. \tag{2,8}$$

It is evident that the induction conditions are satisfied in this case.

If we put in the diagram the new vertex with number V and the new line connected with the first vertex, we obtain

$$(\widetilde{P}_k^l)_V = -\frac{\sum_{j=1}^{V-1} [e_k^j]\, e_l^j}{\det e_V} \qquad \begin{pmatrix} k = 0, 1, ..., V-1 \\ l = V, ..., L \end{pmatrix}. \tag{2,9}$$

$$(\widetilde{P}_k^l)_V = -\frac{\sum_{j=0}^{V-1} [e_k^j]_V\, e_l^j}{\det e_V} \qquad e_l^0 = 0 \;\; (l = V, ..., L). \tag{2,10}$$

From (2,6) we see that

$$[e_k^j]_V = e_0^0\, [e_k^j]_{V-1}, \qquad \det e_V = e_0^0 \det e_{V-1}; \tag{2,11}$$

finally we obtain

$$(\widetilde{P}_k^l)_V = -\frac{\sum_{j=1}^{V-1} [e_k^j]_{V-1}\, e_l^j}{\det e_{V-1}} = (\widetilde{P}_k^l)_{V-1}. \tag{2,12}$$

The conditions of induction are satisfied and therefore the Theorem holds.

The last statement is trivial from physical point of view [4]. The importance of expressions (2,3) and (2,4) lies in the possibility to express all analytic characteristics of the contribution by the matrix of diagram. It permits the application of the above induction method for investigation of analytic properties of the scattering amplitude. The results obtained in this way will be published later. [5].

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THEORETICAL PHYSICS

Space Group of White Tin. I. Symmetry Points

by

M. MIĄSEK and M. SUFFCZYŃSKI

Presented by L. INFELD on April 25, 1961

The white tin structure can be viewed upon as a tetragonal body-centered or face-centered structure with two atoms in the unit cell.

The body-centered model of white tin will be used in this paper. It is exhibited in Figs. 1 and 2. The two lattice constants in white tin are $a = 5.8197$, $c = 3.1749$.

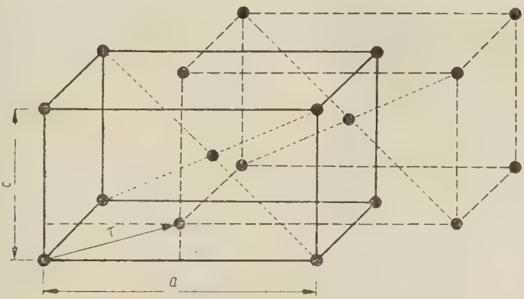

Fig. 1. The body-centered model of the white tin structure

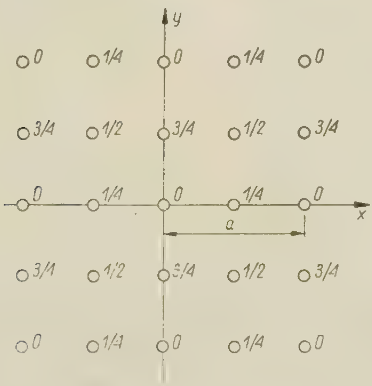

Fig. 2. Projection of the white tin structure on the xy-plane. The heights of the points above this plane are given in units of the lattice constant c

The basic primitive translations of the body-centered lattice Γ_q' are

$$t_1 = (-a/2,\ a/2,\ c/2),$$
$$t_2 = (a/2,\ -a/2,\ c/2),$$
$$t_3 = (a/2,\ a/2,\ -c/2).$$

The origin of the co-ordinate system is taken at one atom, the second atom in the unit cell has the position $\boldsymbol{\tau} = (a/2,\ 0,\ c/4)$.

The space group of white tin is $I\frac{4_1}{a}\frac{2}{m}\frac{2}{d}$ or D_{4h}^{19} [1, 2]. The group of covering operations of the lattice is generated by the generators:

$$[S_4|0],\ [\sigma_{vx}|0],\ [I|\boldsymbol{\tau}],\ [E|\Gamma_q'].$$

Besides the primitive translation operations $[E|\boldsymbol{R}_n]$ we have 16 covering operations. These are enumerated in Table I. We notice that $B_j = B_1 E_j$, $j = 1, ..., 8$.

TABLE I

No.	Notation of Koster [6]	Substitution in x, y, z, co-ordinate system [7]
E_1	$[E\|0]$	$x\,y\,z$
E_2	$[C_2\|0]$	$\bar{x}\,\bar{y}\,z$
E_3	$[S_4\|0]$	$\bar{y}\,x\,\bar{z}$
E_4	$[S_4^3\|0]$	$y\,\bar{x}\,\bar{z}$
E_5	$[\sigma_{vy}\|0]$	$x\,\bar{y}\,z$
E_6	$[\sigma_{vx}\|0]$	$\bar{x}\,y\,z$
E_7	$[C_{2x\bar{y}}''\|0]$	$\bar{y}\,\bar{x}\,\bar{z}$
E_8	$[C_{2xy}''\|0]$	$y\,x\,\bar{z}$
B_1	$[I\|\boldsymbol{\tau}]$	$\frac{1}{2}a - x, \bar{y}, \frac{1}{4}c - z$
B_2	$[\sigma_h\|\boldsymbol{\tau}]$	$\frac{1}{2}a + x, y, \frac{1}{4}c - z$
B_3	$[C_4^3\|\boldsymbol{\tau}]$	$\frac{1}{2}a + y, \bar{x}, \frac{1}{4}c + z$
B_4	$[C_4\|\boldsymbol{\tau}]$	$\frac{1}{2}a - y, x, \frac{1}{4}c + z$
B_5	$[C_{2y}'\|\boldsymbol{\tau}]$	$\frac{1}{2}a - x, y, \frac{1}{4}c - z$
B_6	$[C_{2x}'\|\boldsymbol{\tau}]$	$\frac{1}{2}a + x, \bar{y}, \frac{1}{4}c - z$
B_7	$[\sigma_{dxy}\|\boldsymbol{\tau}]$	$\frac{1}{2}a + y, x, \frac{1}{4}c + z$
B_8	$[\sigma_{dx\bar{y}}\|\boldsymbol{\tau}]$	$\frac{1}{2}a - y, \bar{x}, \frac{1}{4}c + z$

The reciprocal lattice for the body-centered tetragonal structure is given by the basic primitive vectors in the reciprocal space.

$$b_1 = 2\pi\,(0,\ 1/a,\ 1/c),$$
$$b_2 = 2\pi\,(1/a,\ 0,\ 1/c),$$
$$b_3 = 2\pi\,(1/a,\ 1/a,\ 0).$$

The first Brillouin zone of the body-centered tetragonal lattice is shown in Fig. 3, where the points of high symmetry are labelled in agreement with Gold [3]. The

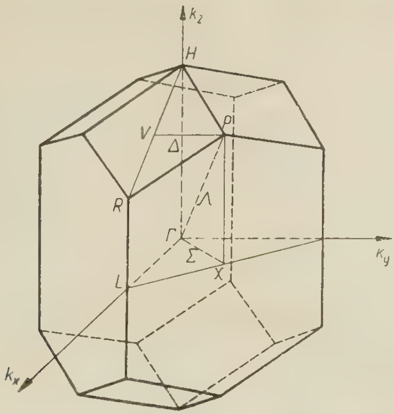

Fig. 3. The first Brillouin zone for the body-centered tetragonal lattice

co-ordinates of the high symmetry points, together with equivalent ones are given in Table II and the operations of the $\boldsymbol{k}$ vector group for each symmetry point are likewise enumerated there.

TABLE II

Symmetry label	Co-ordinates of the $\boldsymbol{k}$ point and the equivalent points	Covering operations transforming $\boldsymbol{k}$ into itself or the equivalent one
Γ	$(0, 0, 0)$	$E_j, B_j \quad j = 1, \ldots, 8.$
L	$(\pm 2\pi/a, 0, 0), (0, \pm 2\pi/a, 0)$	$E_j, B_j \quad j = 1, \ldots, 8.$
X	$(\pi/a, \pi/a, 0), (-\pi/a, -\pi/a, 0)$	$E_1, E_2, E_7, E_8, B_1, B_2, B_7, B_8$
P	$\left\{\begin{array}{l}(\pi/a, \pi/a, \pi/c)\\(-\pi/a, -\pi/a, \pi/c)\\(\pi/a, -\pi/a, -\pi/c)\\(-\pi/a, \pi/a, -\pi/c)\end{array}\right.$	$E_1, E_2, E_3, E_4, B_5, B_6, B_7, B_8$
V	$(\pi/a, 0, \pi/c), (-\pi/a, 0, -\pi/c)$	E_1, E_5, B_1, B_5

The characters of the representations of the space groups for the points of high symmetry in the Brillouin zone have been worked out using a method described by Döring and Zehler [4], Asendorf [5] and Jones [7], [8].

Γ. The centre of the zone has the symmetry of the point group D_{4h}. Its characters are well known [6], [9]. The two-dimensional irreducible representations can be chosen in the form given in Table III.

TABLE III

Representations at the point Γ, $\boldsymbol{k} = (0, 0, 0)$

Elements	$\Gamma_1^{\pm}$	$\Gamma_2^{\pm}$	$\Gamma_3^{\pm}$	$\Gamma_4^{\pm}$	$\Gamma_5^{\pm}$
E_1	1	1	1	1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
E_2	1	1	1	1	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
E_3	1	1	—1	—1	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
E_4	1	1	—1	—1	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
E_5	1	—1	1	—1	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
E_6	1	—1	1	—1	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
E_7	1	—1	—1	1	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
E_8	1	—1	—1	1	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
B_1	± 1	± 1	± 1	± 1	$\pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$B_j = B_1 E_j$ $j = 2, 3, \ldots, 8$					

L. For the wave function at the point L we have $B_7^2 \psi_L = - \psi_L$. Therefore to obtain the group of operations at L we must add to the operations listed in Table II the new element $B_7^2 = T_1 = (a/2+x, a/2+y, c/2+z)$. At the point L, T_1 commutes with every element and T_1^2 is equivalent to identity. Thus at the point L we have to do with a group of order 32.

The irreducible representations at the point L can be chosen in the form given in Table IV. The group of the $\boldsymbol{k}$ vector at L is isomorphic with the group of the $\boldsymbol{k}$ vector at the centre of the square face in the diamond structure [6].

X. For the wave function at X we have $B_2^2 \psi_X = - \psi_X$. Thus, to get the group at the point X we must include the element $B_2^2 = T_2 = (a+x, y, z)$. T_2 commutes with every element and T_2^2 is equivalent to identity. Hence, the group of the $\boldsymbol{k}$ vector at the point X is of order 16. The possible irreducible representations at the point X are given in Table V.

P. For the wave function at the point P we have $B_8^2 \psi_P = \exp(i\pi/2) \psi_P = i\psi_P$. To obtain the group at the point P we must include as new elements: $B_8^2 = T_3 = (a/2+x, -a/2+y, c/2+z)$ and its two powers T_3^2 equivalent to T_2, and T_3^3 equivalent

to T_1. T_3 commutes with every element and T_3^4 is equivalent to identity. Thus the group at the point P is of order 32. The possible irreducible representations at the point P are listed in Table VI. The group of the $\boldsymbol{k}$ vector at the point P is isomorphic with the group of the $\boldsymbol{k}$ vector at the corner point in the diamond structure [6].

TABLE IV

Representations at the point L, $\boldsymbol{k} = (2\pi/a, 0, 0)$

Elements	L_1	L_2	L_3	L_4
E_1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
E_2	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
E_3	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
E_4	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
E_5	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
E_6	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
E_7	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
E_8	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
B_1	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
$B_j = B_1 E_j$ $j = 2, \ldots, 8$				
T_1	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

V. The group of $\boldsymbol{k}$-vector at V is of the order 4, and only one-dimensional irreducible representations exist. These representations are given in Table VII.

The similarities in the symmetry groups of white tin and diamond reflect the fact that the white tin structure is a particular deformation of the diamond structure.

We wish to thank Dr M. L. Glasser for communicating his results for the centre of the zone in white tin.

TABLE V

Representations at the point
X, $k = (\pi/a, \pi/a, 0)$

Elements	X_1	X_2
E_1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
E_2	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
E_7	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
E_8	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
B_1	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$B_j = B_1 E_j$ $j = 2, 7, 8$		
T_2	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

TABLE VI

Representations at the point
P, $k = (\pi/a, \pi/a, \pi/c)$

Elements	P_1	P_2
E_1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
E_2	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
E_3	$\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & i \end{bmatrix}$
E_4	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -i \end{bmatrix}$
B_5	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
B_6	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
B_7	$\begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & i \\ -1 & 0 \end{bmatrix}$
B_8	$\begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix}$
T_1	$\begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$	$\begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$
T_2	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
T_3	$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$	$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

TABLE VII

Representations at the point
V, $k = (\pi/a, 0, \pi/c)$

Elements	V_1	V_2	V_3	V_4
E_1	1	1	1	1
E_5	1	1	—1	—1
B_1	1	—1	1	—1
B_5	1	—1	—1	1

When working on space symmetry of white tin we were unaware of the paper by S. Mase [8].

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THEORETICAL PHYSICS

Space Group of White Tin. II. Symmetry Lines and Planes

by

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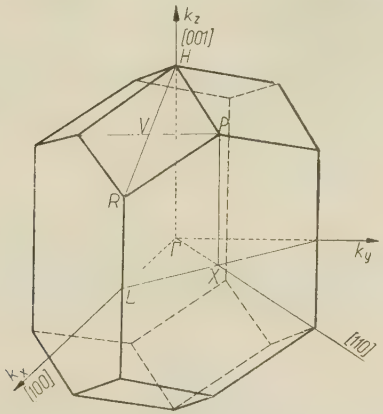
Presented by L. INFELD on April 25, 1961

The first Brillouin zone of the body-centered tetragonal lattice is displayed in the Figure, where the points of high symmetry are labelled as in [1]. The co-ordinate system and the notations adopted here are exactly the same as there.



The first Brillouin zone for the body-centered tetragonal lattice

The co-ordinates of the high symmetry lines and planes, together with their equivalents, are given in Table I, where the operations of the ***k*** vector group for each symmetry line or plane are listed.

The irreducible representations of the single and double groups for the lines and planes of high symmetry are given in Tables II—X. In each Table the additional representations of the double group are listed below the solid line. We always put $\varepsilon = \exp(i\pi/4)$ and denote by an asterisk the complex conjugate.

The space operations with nonvanishing fractional translation, that is the B_j operations [1], are represented by matrices given in the Tables multiplied by the

TABLE I

Symmetry label	Equations of the symmetry lines and symmetry planes	Covering operations transforming $\boldsymbol{k}$ into itself or the equivalent one
ΓH	$k_x = k_y = 0$	$E_1, E_2, E_5, E_6, B_3, B_4, B_7, B_8$
ΓL	$k_y = k_z = 0$	E_1, E_5, B_2, B_6
ΓX	$k_x = k_y,\ k_z = 0$	E_1, E_8, B_2, B_7
LR	$k_x = \pm 2\pi/a,\ k_y = 0;\ k_x = 0,$ $k_y = \pm 2\pi/a;$	$E_1, E_2, E_5, E_6, B_3, B_4, B_7, B_8$
LX	$k_x + k_y = \pm 2\pi/a,\ k_z = 0$	E_1, E_7, B_2, B_8
PX	$k_x = k_y = \pm \pi/a$	E_1, E_2, B_7, B_8
PV	$k_x = \pi/a,\ k_z = \pi/c;$ $k_x = -\pi/a,\ k_z = -\pi/c;$	E_1, B_5
ΓPX	$k_x = k_y$	E_1, B_7
ΓLR	$k_y = 0$	E_1, E_5
LPR	$k_x + k_y = \pm 2\pi/a$	E_1, B_8

phase factor $\gamma = \exp(i\boldsymbol{k}\boldsymbol{R}_n/2)$, specified in the headline of the Table. $\boldsymbol{k}$ is the wave-vector under consideration and $\boldsymbol{R}_n$ is one of the three primitive vectors

$$
\begin{aligned}
\boldsymbol{R}_1 &= \boldsymbol{t}_1 + \boldsymbol{t}_2 + \boldsymbol{t}_3 &&= a/2,\ a/2,\ c/2, \\
\boldsymbol{R}_2 &= \boldsymbol{t}_2 + \boldsymbol{t}_3 &&= a,\ 0,\ 0, \\
\boldsymbol{R}_3 &= \boldsymbol{t}_2 &&= a/2,\ -a/2,\ c/2.
\end{aligned}
$$

Here $\boldsymbol{t}_1$, $\boldsymbol{t}_2$, $\boldsymbol{t}_3$ are the basic primitive translations chosen in [1].

The Tables II—IV contain the irreducible representations of the lines within the Brillouin zone.

TABLE II

The line ΓH; $k_x = k_y = 0$; $\gamma = \exp(i\,\boldsymbol{k}\boldsymbol{R}_1/2)$

	E_1	E_2	E_5	E_6	B_3	B_4	B_7	B_8
1	1	1	1	1	1	1	1	1
2	1	1	1	1	—1	—1	—1	—1
3	1	1	—1	—1	1	1	—1	—1
4	1	1	—1	—1	—1	—1	1	1
5	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}-1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1&0\\0&-1\end{bmatrix}$	$\begin{bmatrix}-1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&-1\\1&0\end{bmatrix}$	$\begin{bmatrix}0&1\\-1&0\end{bmatrix}$	$\begin{bmatrix}0&-1\\-1&0\end{bmatrix}$	$\begin{bmatrix}0&1\\1&0\end{bmatrix}$
$6^\pm$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}i&0\\0&-i\end{bmatrix}$	$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	$\begin{bmatrix}0&-i\\-i&0\end{bmatrix}$	$\pm\begin{bmatrix}\varepsilon&0\\0&\varepsilon^*\end{bmatrix}$	$\pm\begin{bmatrix}-\varepsilon^*&0\\0&-\varepsilon\end{bmatrix}$	$\pm\begin{bmatrix}0&\varepsilon^*\\-\varepsilon&0\end{bmatrix}$	$\pm\begin{bmatrix}0&-\varepsilon\\\varepsilon^*&0\end{bmatrix}$

The group of the line $\Gamma\pm$ is isomorphic with the group C_{4v} [2]. The representations 6^+ and 6^- are degenerate as a consequence of the time reversal symmetry.

TABLE III

The line ΓL; $k_y = k_z = 0$; $\gamma = \exp(i\,\boldsymbol{kR}_2/2)$

	E_1	E_5	B_2	B_6
1	1	1	1	1
2	1	1	−1	−1
3	1	−1	1	−1
4	1	−1	−1	1
5	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&1\\-1&0\end{bmatrix}$	$\begin{bmatrix}i&0\\0&-i\end{bmatrix}$	$\begin{bmatrix}0&-i\\-i&0\end{bmatrix}$

TABLE IV

The line ΓX; $k_x = k_y$; $\gamma = \exp(i\,\boldsymbol{kR}_1/2)$

	E_1	E_8	B_2	B_7
1	1	1	1	1
2	1	1	−1	−1
3	1	−1	1	−1
4	1	−1	−1	1
5	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&-\varepsilon\\\varepsilon^*&0\end{bmatrix}$	$\begin{bmatrix}i&0\\0&-i\end{bmatrix}$	$\begin{bmatrix}0&\varepsilon^*\\-\varepsilon&0\end{bmatrix}$

TABLE V

The line LX; $k_x+k_y = 2\pi/a$, $k_z = 0$; $\gamma = \exp(i\boldsymbol{kR}_2/2)$

	E_1	E_7	B_2	B_8	$[E\mid\boldsymbol{R}_1]$
1	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1&0\\0&-1\end{bmatrix}$	$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	$\begin{bmatrix}0&-1\\1&0\end{bmatrix}$	$\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$
2	1	i	i	i	−1
3	1	i	−i	−1	−1
4	1	−i	i	−1	−1
5	1	−i	i	i	−1

The representations 2, 3 and 4, 5 are degenerate as a consequence of the time reversal symmetry.

The groups for the lines ΓL and ΓX are isomorphic with the group C_{2v} [2].

The Tables V—VII list the irreducible representations of the lines on the boundary of the Brillouin zone.

The line LR; $k_x = 2\pi/a$, $k_y = 0$, is isomorphic with the group of the line of ΓH. The phase factor is $\gamma = \exp(i\boldsymbol{k}\boldsymbol{R}_1/2)$ with $\boldsymbol{k}$ appropriate for the LR line.

TABLE VI

The line PX; $k_x = k_y = \pi/a$; $\gamma = \exp(i\boldsymbol{k}\boldsymbol{R}_3/2)$

	E_1	E_2	B_7	B_8	$[E\|\boldsymbol{R}_2]$
1	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$	$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$	$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$	$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$
2	1	i	1	i	—1
3	1	i	—1	—i	—1
4	1	—i	—1	i	—1
5	1	—i	1	i	—1

The representations 2, 3 and 4,5 are degenerate as a consequence of the time reversal symmetry.

TABLE VII

The line PV; $k_x = \pi/a$, $k_z = \pi/c$; $\gamma = 1$

	E_1	B_5
1	1	1
2	1	—1
3	1	i
4	1	—i

The Tables VIII—X list the irreducible representations for the planes of symmetry in the Brillouin zone. All groups here are isomorphic with the group C_s [2].

TABLE VIII

The plane ΓPX; $k_x = k_y$; $\gamma = \exp(i\boldsymbol{k}\boldsymbol{R}_1/2)$

	E_1	B_7
1	1	1
2	1	—1
3	1	i
4	1	—i

TABLE IX

The plane ΓLR; $k_y = 0$;

	E_1	E_5
1	1	1
2	1	—1
3	1	i
4	1	—i

TABLE X

The plane LPR; $k_x + k_y = 2\pi/a$; $\gamma = \exp(i\boldsymbol{k}\boldsymbol{R}_3/2)$

	E_1	B_8
1	1	1
2	1	—1
3	1	i
4	1	—i

At a point of no special symmetry in $\boldsymbol{k}$ space the additional representations are always doubly degenerate, because white tin has a centre of inversion, like the diamond.

The compatibility relations are listed for the points Γ, L and X in the Table XI. The numbers in a columns denote the different representations of a symmetry line specified at the top.

TABLE XI

Compatibility relations of the single groups connecting symmetry points with symmetry lines

	ΓL	ΓH	ΓX		ΓL	LR	LX		ΓX	LX	PX
Γ_1^+	1	1	1	L_1	3,4	3,4	1	X_1	1,2	1	1
Γ_2^+	3	3	3	L_2	1,2	1,2	1	X_2	3,4	1	1
Γ_3^+	1	2	3	L_3	1,4	5	1				
Γ_4^+	3	4	1	L_4	2,3	5	1				
Γ_5^+	2,4	5	2,4								
Γ_1^-	2	2	2								
Γ_2^-	4	4	4								
Γ_3^-	2	1	4								
Γ_4^-	4	3	2								
Γ_5^-	1,3	5	1,3								

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THEORETICAL PHYSICS

Space Group of White Tin. III. Double Group

by

M. SUFFCZYŃSKI

Presented by L. INFELD on May 2, 1961

The effects of the spin-orbit coupling become pronounced for elements of high atomic number. Therefore in an analysis of the space group of white tin it seems reasonable to include also the irreducible representations of the double group.

The co-ordinate system and notations used here are the same as in [1], [2].

A proper rotation may be specified by the Euler angles α, β, γ, with $0 \leqslant \alpha < 2\pi$, $0 \leqslant \beta \leqslant \pi$, $0 \leqslant \gamma < 2\pi$. We use the convention adopted by Lomont [3]. The rotation matrix acting on the vector x, y, z in configuration space is a product of three matrices

$$(1)\qquad R_{z''}(\alpha)\, R_{y'}(\beta)\, R_z(\gamma) = \begin{bmatrix} \cos\alpha, & \sin\alpha, & 0 \\ -\sin\alpha, & \cos\alpha, & 0 \\ 0, & 0, & 1 \end{bmatrix} \begin{bmatrix} \cos\beta, & 0, & -\sin\beta \\ 0, & 1, & 0 \\ \sin\beta, & 0, & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma, & \sin\gamma, & 0 \\ -\sin\gamma, & \cos\gamma, & 0 \\ 0, & 0, & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma, & \cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma, & -\cos\alpha\sin\beta \\ -\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\gamma, & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma, & \sin\alpha\sin\beta \\ \sin\beta\cos\gamma, & \sin\beta\sin\gamma, & \cos\beta \end{bmatrix}.$$

The corresponding matrix acting on the spinor column $\begin{pmatrix} u \\ v \end{pmatrix}$

$$(2)\qquad D^{(1/2)}_{z''}(\alpha)\, D^{(1/2)}_{y'}(\beta)\, D^{(1/2)}_{z}(\gamma) = \pm \begin{bmatrix} e^{i\alpha/2}, & 0 \\ 0, & e^{-i\alpha/2} \end{bmatrix} \begin{bmatrix} \cos\beta/2, & \sin\beta/2 \\ -\sin\beta/2, & \cos\beta/2 \end{bmatrix} \begin{bmatrix} e^{i\gamma/2}, & 0 \\ 0, & e^{-i\gamma/2} \end{bmatrix}$$

$$= \pm \begin{bmatrix} e^{i(\alpha+\gamma)/2}\cos\beta/2, & e^{i(\alpha-\gamma)/2}\sin\beta/2 \\ -e^{-i(\alpha-\gamma)/2}\sin\beta/2, & e^{-i(\alpha+\gamma)/2}\cos\beta/2 \end{bmatrix}$$

is determined by (1) up to the sign.

The covering operations of the white tin lattice have the Euler angle β either 0 or π. Hence, only the combination $\alpha+\gamma$ or $\alpha-\gamma$ is determined, and not the angles α and γ separately. For inhomogeneous operations $B_j = B_1 E_j$ ($j = 1, 2, ..., 8$), see [1]. The operation B_1 consists of an inversion $[I|\tau]$. Thus, only the list of E_j and B_1 is essential. For completeness all covering operations of white tin are listed in Table I, together with their Eulerian angles and the spin matrices.

TABLE I

Operation	Notation of Koster [4]	Substitution in Cartesian co-ordinates [5]	Matrix in configuration space	Euler angles	Matrix in spin space
E_1	$[E\|0]$	$x\,y\,z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\beta = 0,\ \alpha+\gamma = 0,$	$D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
E_2	$[C_2\|0]$	$\bar{x}\,\bar{y}\,z$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$0,\ \pi,$	$D_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$
E_3	$[S_4\|0]$	$\bar{y}\,x\,\bar{z}$	$\mathrm{I}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$0,\ \pi/2,$	$D_3 = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^* \end{bmatrix}$
E_4	$[S_4^3\|0]$	$y\,\bar{x}\,\bar{z}$	$\mathrm{I}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$0,\ 3\pi/2,$	$D_4 = \begin{bmatrix} -\varepsilon^* & 0 \\ 0 & -\varepsilon \end{bmatrix}$
E_5	$[\sigma_{vy}\|0]$	$x\,\bar{y}\,z$	$\mathrm{I}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\beta = \pi,\ \gamma-\alpha = 0,$	$D_5 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
E_6	$[\sigma_{vx}\|0]$	$\bar{x}\,y\,z$	$\mathrm{I}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\pi,\ \pi,$	$D_6 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$
E_7	$[C_{2x\bar{y}}\|0]$	$\bar{y}\,\bar{x}\,\bar{z}$	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\pi,\ \pi/2,$	$D_7 = \begin{bmatrix} 0 & \varepsilon^* \\ -\varepsilon & 0 \end{bmatrix}$
E_8	$[C''_{2xy}\|0]$	$y\,x\,\bar{z}$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\pi,\ 3\pi/2$	$D_8 = \begin{bmatrix} 0 & -\varepsilon \\ \varepsilon^* & 0 \end{bmatrix}$

B_1	$[I\|\boldsymbol{\tau}]$	$\frac{1}{2}a - x, \bar{y}, \frac{1}{4}c - z$	$\begin{bmatrix} -1 & 0 & 0 & a/2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & c/4 \end{bmatrix}$	$\beta = 0,\ a+\gamma = 0,$	D_1
B_2	$[\sigma_h\|\boldsymbol{\tau}]$	$\frac{1}{2}a + x, y, \frac{1}{4}c - z$	$\mathrm{I}\begin{bmatrix} -1 & 0 & 0 & a/2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & c/4 \end{bmatrix}$		D_2
B_3	$[C_4^3\|\boldsymbol{\tau}]$	$\frac{1}{2}a + y, \bar{x}, \frac{1}{4}c + z$	$\begin{bmatrix} 0 & 1 & 0 & a/2 \\ -1 & 0 & 0, & 0 \\ 0 & 0 & 1 & c/4 \end{bmatrix}$		D_3
B_4	$[C_4\|\boldsymbol{\tau}]$	$\frac{1}{2}a - y, x, \frac{1}{4}c + z$	$\begin{bmatrix} 0 & -1 & 0 & a/2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & c/4 \end{bmatrix}$		D_4
B_5	$[C'_{2y}\|\boldsymbol{\tau}]$	$\frac{1}{2}a - x, y, \frac{1}{4}c - z$	$\begin{bmatrix} -1 & 0 & 0 & a/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & c/4 \end{bmatrix}$		D_5
B_6	$[C'_{2x}\ \boldsymbol{\tau}]$	$\frac{1}{2}a + x, \bar{y}, \frac{1}{4}c - z$	$\begin{bmatrix} 1 & 0 & 0 & a/2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & c/4 \end{bmatrix}$		D_6
B_7	$[\sigma_{dxy}\|\boldsymbol{\tau}]$	$\frac{1}{2}a + y, x, \frac{1}{4}c + z$	$\mathrm{I}\begin{bmatrix} 0 & -1 & 0 & a/2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & c/4 \end{bmatrix}$		D_7
B_8	$[\sigma_{dx\bar{y}}\|\boldsymbol{\tau}]$	$\frac{1}{2}a - y, \bar{x}, \frac{1}{4}c + z$	$\mathrm{I}\begin{bmatrix} 0 & 1 & 0 & a/2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & c/4 \end{bmatrix}$		D_8

TABLE II

Additional representations of the double group at the point Γ, $\boldsymbol{k} = (000)$

Operation	$\Gamma_6^\pm$	$\Gamma_7^\pm$
E_1	D_1	D_1
E_2	D_2	D_2
E_3	D_3	$-D_3$
E_4	D_4	$-D_4$
E_5	D_5	D_5
E_6	D_6	D_6
E_7	D_7	$-D_7$
E_8	D_8	$-D_8$
B_1	$\pm D_1$	$\pm D_1$
$B_j = B_1 E_j$ $j = 2, 3, \ldots, 8.$		

$$\Gamma_1^\pm \times D^{1/2} = \Gamma_2^\pm \times D^{1/2} = \Gamma_6^\pm$$

$$\Gamma_3^\pm \times D^{1/2} = \Gamma_4^\pm \times D^{1/2} = \Gamma_7^\pm$$

$$\Gamma_5^\pm \times D^{1/2} = \Gamma_6^\pm + \Gamma_7^\pm$$

TABLE III

Additional representations of the double group at the point L, $\boldsymbol{k} = (2\pi/a, 0, 0)$

Operation	L_5
E_1	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
E_2	$\begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$
E_3	$\begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon^* & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 \\ 0 & 0 & 0 & -\varepsilon^* \end{bmatrix}$
E_4	$\begin{bmatrix} -\varepsilon^* & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon^* & 0 \\ 0 & 0 & 0 & \varepsilon \end{bmatrix}$
E_5	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$
E_6	$\begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix}$
E_7	$\begin{bmatrix} 0 & \varepsilon^* & 0 & 0 \\ -\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon^* \\ 0 & 0 & \varepsilon & 0 \end{bmatrix}$
E_8	$\begin{bmatrix} 0 & -\varepsilon & 0 & 0 \\ \varepsilon^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & -\varepsilon^* & 0 \end{bmatrix}$
B_1	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$
$B_j = B_1 E_j$ $j = 2, 3, \ldots, 8$	
$B_7^2 = \overline{T}_1$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$L_r \times D^{1/2} = L_5, \quad r = 1, 2, 3, 4.$$

TABLE IV

Additional representations of the double group at the point X, $\boldsymbol{k} = (\pi/a,\ \pi/a,\ 0)$

Operation	$X_3^\pm$
E_1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
E_2	$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$
E_7	$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$
E_8	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
B_1	$\pm \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
$B_j = B_1 E_j$ $j = 2, 7, 8.$	
$B_2^2 = \overline{T}_2$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$X_1 \times D^{1/2} = X_2 \times D^{1/2} = X_3^+ + X_3^-$$

The representations $X_3^\pm$ are degenerate as a consequence of time reversal symmetry.

TABLE V

Additional representations of the double group at the point P, $k = (\pi/a, \pi/a, \pi/c)$

Operation	P_3	P_4	P_5	P_6	P_7
E_1	1	1	1	1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
E_2	$-i$	$-i$	$-i$	$-i$	$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$
E_3	ε^*	$-\varepsilon^*$	ε^*	$-\varepsilon^*$	$\begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix}$
E_4	$-\varepsilon$	ε	$-\varepsilon$	ε	$\begin{bmatrix} -\varepsilon^* & 0 \\ 0 & \varepsilon^* \end{bmatrix}$
B_5	i	i	$-i$	$-i$	$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$
B_6	1	1	-1	-1	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
B_7	ε	$-\varepsilon$	$-\varepsilon$	ε	$\begin{bmatrix} 0 & \varepsilon^* \\ -\varepsilon^* & 0 \end{bmatrix}$
B_8	ε^*	$-\varepsilon^*$	$-\varepsilon^*$	ε^*	$\begin{bmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{bmatrix}$
$B_7^2 = \overline{T}_1$	i	i	i	i	$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$
$B_6^2 = \overline{T}_2$	1	1	1	1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$B_1^2 = \overline{T}_3$	$-i$	$-i$	$-i$	$-i$	$\begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$

$$P_1 \times D^{1/2} = P_3 + P_5 + P_7 \qquad P_2 \times D^{1/2} = P_4 + P_6 + P_7$$

The presentations P_3, P_5 and P_4, P_6 are degenerate as a consequence of time reversal symmetry.

TABLE VI

Additional representations of the double group at the point V, $k = (\pi/a, 0, \pi/c)$

Operation	V_5	V_6	V_7	V_8
E_1	1	1	1	1
E_5	i	i	$-i$	$-i$
B_1	1	-1	-1	1
B_5	i	$-i$	i	$-i$

$$V_1 \times D^{1/2} = V_3 \times D^{1/2} = V_5 + V_8, \qquad V_2 \times D^{1/2} = V_4 \times D^{1/2} = V_6 + V_7$$

The representations V_5, V_7 and V_6, V_8 are degenerate as a consequence of time reversal symmetry.

TABLE VII

Compatibility relations for the additional representations of the double groups connecting symmetry points with symmetry lines

Γ	ΓH	X	LX	PX	L	LX	LR
$\Gamma_6^{\pm}$	$6^{\pm}$	X_3^+	2+5	2+4	L_5	2+3+4+5	$6^+ + 6^-$
$\Gamma_7^{\pm}$	$6^{\mp}$	X_3^-	3+4	3+5			

P	PX	PR	PV	PH
P_3	5	4	3	4
P_4	4	3	3	3
P_5	4	3	4	3
P_6	5	4	4	4
P_7	2+3	3+4	3+4	3+4

R	LR	PR	RV	H	HP	HR	V	VP	VR
R_6^+	6^+	3+4	3+4	H_6^+	3+4	3+4	V_5	3	3
R_6^-	6^-	3+4	3+4	H_6^-	3+4	3+4	V_6	4	3
							V_7	3	4
							V_8	4	4

The additional irreducible representations of the double group at the points of high symmetry are listed in Tables II—VI. In the Tables $\varepsilon = \exp i\pi/4$ and the asterisk denotes the complex conjugate.

The barred translation operations used in the Tables correspond to the three non-coplanar primitive translations introduced in [1], [2].

$$(3) \qquad \begin{cases} \overline{T}_1 = [\overline{E}|\boldsymbol{R}_1], & \boldsymbol{R}_1 = a/2,\ a/2,\ c/2, \\ \overline{T}_2 = [\overline{E}|\boldsymbol{R}_2], & \boldsymbol{R}_2 = a,\ 0,\ 0, \\ \overline{T}_3 = [\overline{E}|\boldsymbol{R}_3], & \boldsymbol{R}_3 = a/2,\ -a/2,\ c/2. \end{cases}$$

The compatibility relations connecting the representations of the single group with the additional representations of the double group are given in each Table.

The compatibility relations for the additional representations of the double groups connecting high symmetry points with symmetry lines [2] are listed

in Table VII if they are not trivial in the sense used by Elliott [6]. The numbers in each column denote the irreducible representations of the symmetry line specified at the top of the column.

I should like to thank Dr. M. Lawrance Glasser for communicating his results on the symmetry at the zone centre in white tin.

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EXPERIMENTAL PHYSICS

Irreversible Photobleaching of Acriflavine in Polyvinyl Alcohol

by

J. HELDT

Presented by A. JABŁOŃSKI on March 18, 1961

As observed by various authors [1]—[3], the excitation energy of dye molecules is by no means in all cases emitted in the form of a light *quantum* (fluorescence, phosphorescence, or slow fluorescence); but can also serve as activation energy in photochemical reactions. Observable changes in the absorption and emission spectra [1]—[4] as well as in emission anisotropy [5] occur after the dye has been illuminated for a rather long period of time from a source of considerable light intensity. Certain dyes, as, e.g., acridine yellow [4]—[5] and methyl orange exhibit reversible changes, whereas others, as, e.g., uranin, eosin, fluorescein [2]—[3] and chlorophyll in alcohol solutions at room temperature and at — 183°C undergo irreversible photochemical changes.

The purpose of this work was to investigate the changes in acriflavine in a rigid medium when illuminated with visible light at room temperature.

As solvent polyvinyl alcohol polymer was used. The dye was recrystallized and its purity tested by paper chromatography. Acriflavine organophosphors of various concentrations in polyvinyl alcohol were prepared by pouring solutions containing different concentrations of the dye onto glass plates washed thoroughly and dried. Polymerization of the aqueous solution of polyvinyl alcohol lasted 5 days in the dark at 20°C. The thickness of all phosphors (films) was the same (0.07 mm.), as verified by measuring with a gauge. The absorption spectra of the phosphors were measured with a Zeiss universal spectrophotometer. The phosphors were illuminated through a water-glass filter with light from a high-pressure HBO-200 mercury lamp supplied by a 100 V battery. Illumination lasted 15 or 20 min., after which the spectrum of the illuminated and that of the non-illuminated phosphor were measured simultaneously.

Figs. 1, 2 and 3 show the dependence of the extinction coefficient on the wavelength of the light absorbed and on the duration of illumination, for three different concentrations. The areas P bounded by the absorption curves were measured with a planimeter. The values of P, in arbitrary units, for the various concentrations and times of illumination are given in the Table. The dependence of the area P on the time of illumination is shown in Fig. 4, where the scale used for the ordinates

of graphs 1 and 2 is the same, whereas that of graph 3, corresponding to the concentration $c = 2.29 \times 10^{-5}$ mol./l., is three times larger.

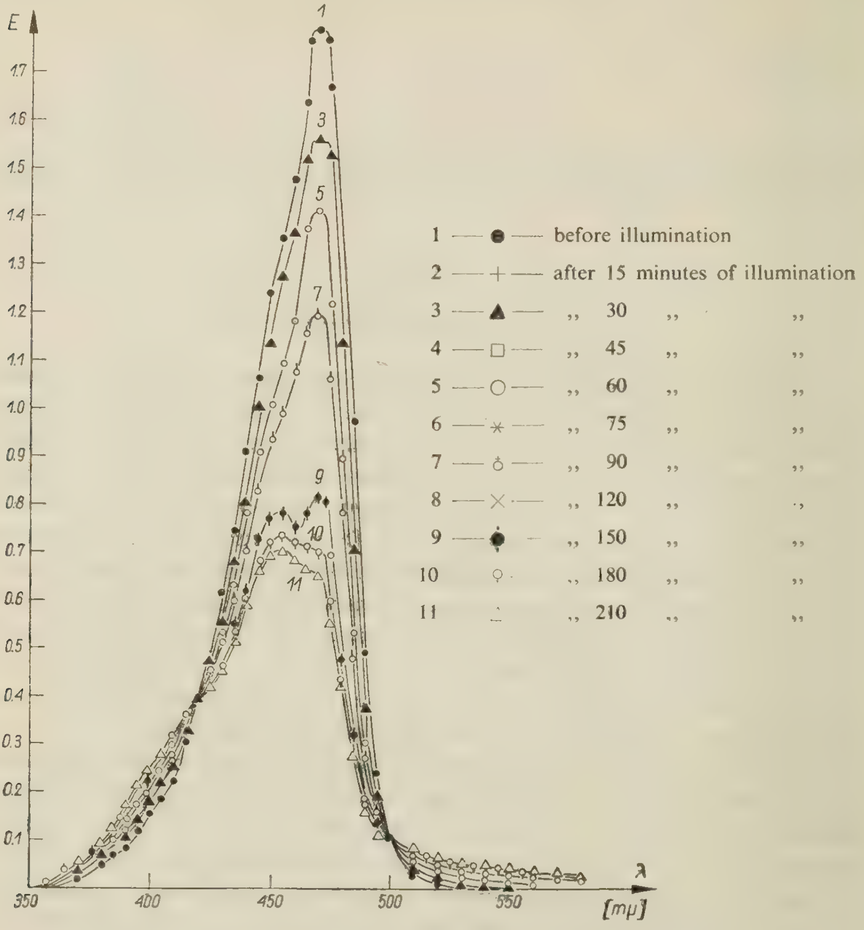

Fig. 1. Absorption spectrum of acriflavine in polyvinyl alcohol. Concentration of dye $c = 4.86 \times 10^{-4}$ mol./l.

In view of the fact that the oscillator strength of the absorbing molecules is the same before and after illumination, the expression

$$c = c_0 \frac{\int_0^\infty \varepsilon_\nu \, d\nu}{\int_0^\infty \varepsilon_{\nu 0} \, d\nu}, \tag{1}$$

derived from the generalized Porter-Windsor formula [7] for the oscillator strength, could be used for evaluating the concentration of absorbing molecules after 30 mi-

TABLE

Dependence of P and kI_0 on concentration of dye and duration of illumination

Duration of illumination (minutes)	P area in arbitrary units												C calc., after 30 min. illumination, in mol./l.	kI_0, calc., $\times 10^{-11}$
	0	15	30	45	60	75	90	120	150	180	210	240		
$c = 4.86 \times 10^{-4}$ mol./l.	79.5		65.4		58.5		53.3		44.8	41.6	40.9	38.9	3.99×10^{-4}	3.29
$c = 8.61 \times 10^{-5}$ mol./l.	40.4	33.0	29.5	26.1	22.8		19.8	18.2	17.0	16.3			6.33×10^{-5}	3.78
$c = 2.29 \times 10^{-5}$ mol./l.	9.41	7.02	5.80	5.46		4.55							1.41×10^{-5}	4.37
$c = 1.86 \times 10^{-5}$ mol./l.	8.19	5.60	4.86	3.51									1.10×10^{-5}	4.67

nutes of illumination (here c_0 and c are the concentrations of the absorbing molecules before and after 30 min. of illumination, respectively, and $\varepsilon_{\nu 0}$, ε_ν are the respective molar extinction coefficients). The molecular concentrations of the dye subsequent to illumination, as computed from (1), are given in the third column of the Table.

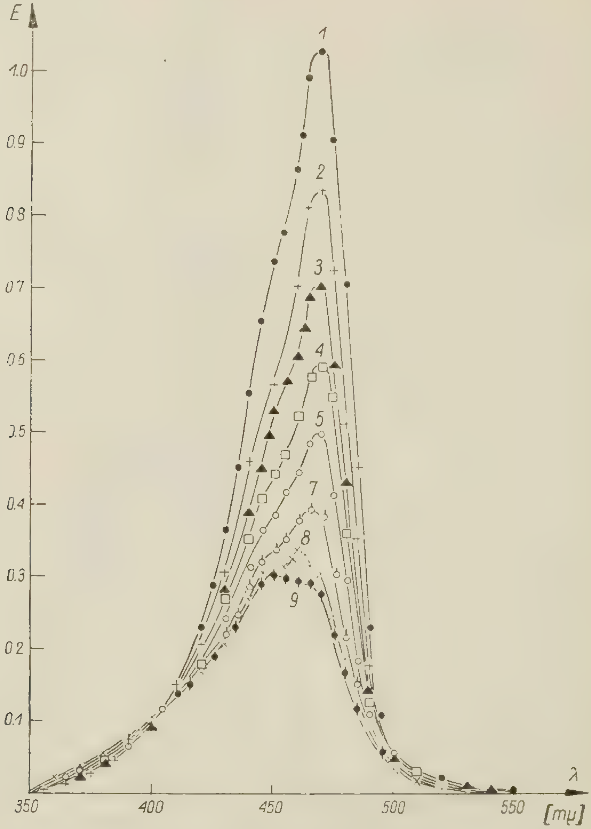

Fig. 2. Absorption spectrum of acriflavine in polyvinyl alcohol. Concentration of dye $c = 8.61 \times 10^{-5}$ mol./l. For explanations see Fig. 1.

Integration of the Van't Hoff equation [8] describing the time variations of the substrates of the photoreaction yields the following equation:

$$\varepsilon l (c_0 - c) + \ln \frac{1 - e^{-\varepsilon c_0 l}}{1 - e^{-\varepsilon c l}} = \varepsilon l k I_0 t , \tag{2}$$

where I_0 denotes the radiation intensity of the mercury lamp, l — the thickness of the phosphor, t — the duration of illumination, k — a constant proportional to the quantum yield of the photoreaction, whereas ε and c are the same as in (1).

From (2), the product kI_0 was computed; the values are given in the Table and plotted *vs.* the concentration in Fig. 5.

In order to establish whether the process of photobleaching of the dye is a reversible one, the phosphor was heated during an hour at 40°C. The absorption spectra of non-illuminated, illuminated and those of illuminated and heated phosphors reveal different extinction coefficients, as shown in Fig. 6.

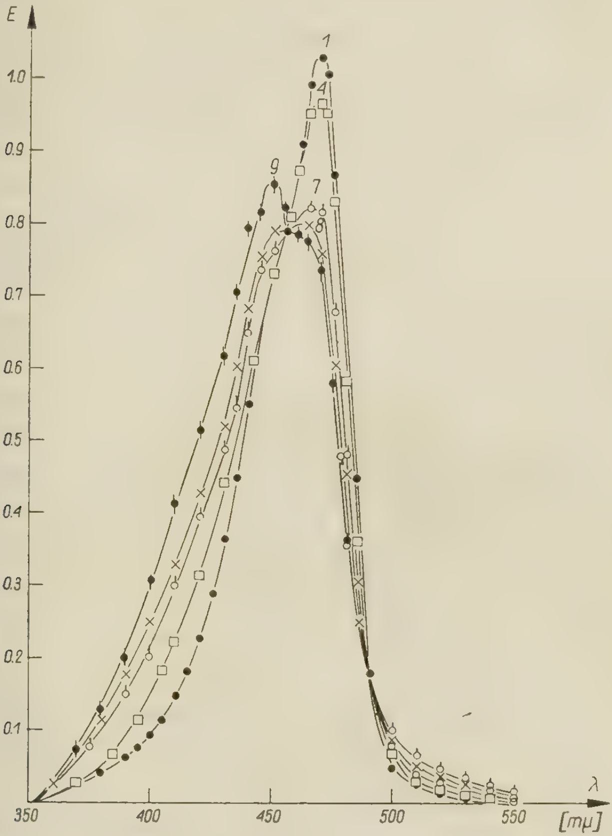

Fig. 2a. Absorption spectra of illuminated phosphor as normalized to non-illuminated phosphor. The notation is that of Fig. 2

Discussion of results

From Figs. 1 and 2, acriflavine in polyvinyl alcohol is seen to form associates at concentrations of the dye as low as 8.61×10^{-5} mol./l.; it would seem that these associates account for the distinct maximum in the absorption band at $\lambda = 450$ mμ.

The light quantum absorbed by the dye causes transition of the molecule into the excited state F [16], whence it can either return to the ground state N with emission of fluorescent light or proceed to the metastable state M by radiationless transition. The mean life of the molecule in the metastable state is of the order of seconds, providing for a high probability of a photochemical reaction, which reveals itself

in changes in the absorption spectrum (Figs. 1—3), in the emission spectrum [4] and in other features of luminescence [5], [8].

Let A denote a dye molecule in state N, A^* and A^{**} — molecules in the excited states F and M, and R — a polyvinyl alcohol radical. Illumination of acriflavine leads to excitation of the molecule:

$$A \xrightarrow{h\nu} A^* \rightarrow A^{**}.$$

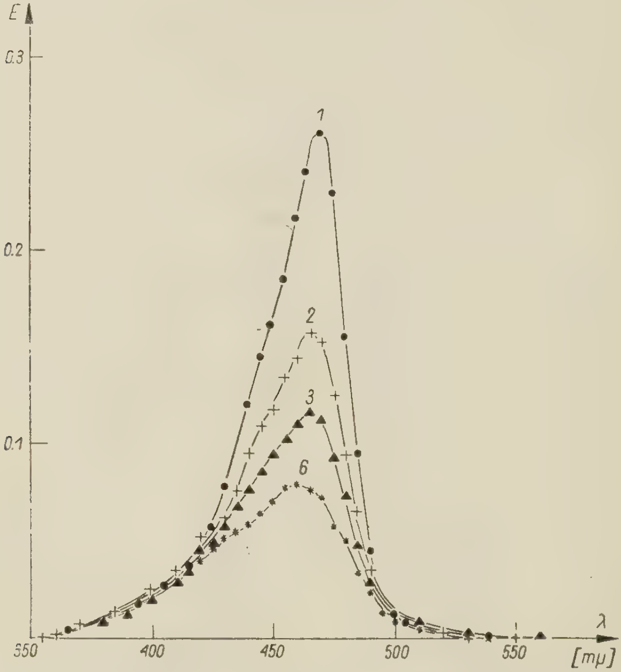

Fig. 3. Absorption spectrum of acriflavine in polyvinyl alcohol. Concentration of dye $c = 2.29 \times 10^{-5}$ ml./l. For explanations see Fig. 1

Acriflavine in the excited states can react with the surrounding medium according to formulae which, in the case under consideration, fall in three groups:

Group 1.

(a) $A^{**} + RH \rightarrow AH^{\oplus} + R^{\ominus}$

(b) $AH^{\oplus} + AH^{\oplus} \rightarrow AH_2 + A$

(c) $R^{\ominus} + R^{\ominus} \rightarrow RH + \text{aldehyde}$

The reaction of $AH^{\oplus}$ with the hydroxylic group of water present in the polyvinyl alcohol is of probability equal to that of (b):

(d) $AH^{\oplus} + OH^{\ominus} \rightarrow AH\,(OH)$.

The products of the reactions (b) and (d) correspond to lacton forms of acriflavine possessing an absorption spectrum in the ultraviolet.

Group 2. The reactions consist in the formation of a hydrogen bond between the dye molecule and water molecule present in the polyvinyl alcohol, as well as with oxygen from atmospheric air.

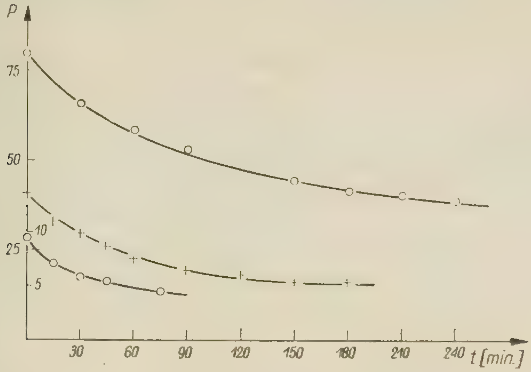

Fig. 4. Dependence of P on dye concentration and duration of illumination

Group 3. The photoreactions do not give rise to bleaching, though they modify the absorption spectrum of the dye. They consist in the formation of associates in state F:

$$A^* + A \rightarrow (AA)^* \xrightarrow[\text{transition}]{\text{radiationless}} (AA).$$

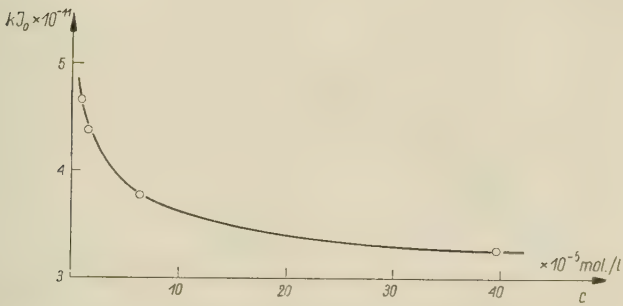

Fig. 5. Relative yield of photobleaching reaction *versus* dye concentration

The increase in the extinction coefficient of the phosphor subsequent to heating (cf. Fig. 6) should be accounted for by a loss of water from the medium, leading to partial reversibility of the reaction of bleaching (Group 1 (d), and 2). Further heating ceases to affect the absorption of acriflavine.

The number of associates produced in the initial stages of the photoreaction is seen from Fig. 2a, to be proportional to the duration of illumination of the phosphor. By further illumination, however, the rate of production of dimers de-

clines, and their number reaches finally a limiting value corresponding to a state of equilibrium between the number of associates produced and dissociated during illumination. Equilibrium sets in after different periods of time for different concentrations of the phosphor (after 2.5 hours in a phosphor of dye concentration $c = 8.61 \times 10^{-5}$ mol./l., and 3 hours in that of $c = 4.86 \times 10^{-4}$ mol./l.).

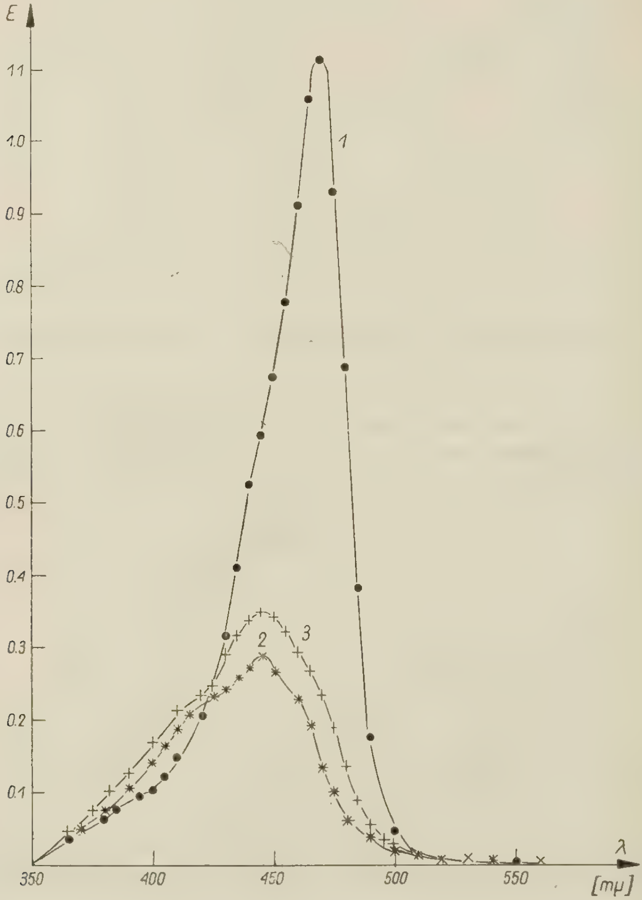

Fig. 6. Absorption spectra of acriflavine in polyvinyl alcohol:
1 —●— non-illuminated, 2 —✱— illuminated during 240 minutes,
3 —+— illuminated and heated

The decrease (Fig. 5) in the quantum yield of the photoreaction with dye concentration is to be explained by the increase in the number of associates and by the shortening of the life of the molecule in metastable state M.

For this reason, Eq. (2) can be applied for the description of the initial stages of the reaction of photobleaching of acriflavine in polyvinyl alcohol only.

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EXPERIMENTAL PHYSICS

Influence of the Concentration of the Solute and of the Polarity of the Solvent on the Absorption Spectra of Fluorescein

by

A. GUTSZE and H. WALERYŚ

Presented by A. JABŁOŃSKI on April 4, 1961

Apart from ionic forms — cations and bivalent anions (according to the *pH*-value of the solution) [1]—[5] — fluorescein has two neutral modifications, namely the leuco- and the quinoid modification [1], [4]. Both these neutral modifications occur in solid state and, depending on the method of their crystallization, either the yellow-leuco or the red-quinoid modification may be obtained. In most solvents these modifications transform immediately into one of the ionic forms with characteristic absorption bands in the visible region. Only in few media fluorescein may partially exist in neutral form.

The aim of the present paper is to investigate, by means of absorption spectra, the influence of the polarity of the medium and that of the fluorescein concentration on the behaviour of particular fluorescein forms in solution.

The polarity was varied by applying

1. various solvents, viz. acetone and methanol,

2. binary mixtures of methanol and non-polar CCl_4, which does not dissolve fluorescein.

The dye solutions were prepared in the following ways:

1. the dye was dissolved in methanol to various concentrations,

2. methanol was dissolved in CCl_4, the volume ratios being 1:1, 1:5, 1:25, 1 : 100; in this way the polarity of the solution was changed, yet the same fluorescein concentration was used in all these solutions;

3. while maintaining constant fluorescein concentration in methanol, CCl_4 was added, to obtain the volume ratios of methanol and CCl_4 as given above. The immediate neighbourhood of fluorescein remains the same in different solutions of this kind, whereas the interaction between dye molecules does change.

All the above solutions were prepared with yellow as well as with red fluorescein. Measurements were carried out by means of a Zeiss spectrophotometer at room temperature.

Figs. 1a and 1b show the absorption spectra of yellow and red fluorescein in methanol for concentrations ranging from 6×10^{-9} to 6×10^{-7} mol./cm^3. In these

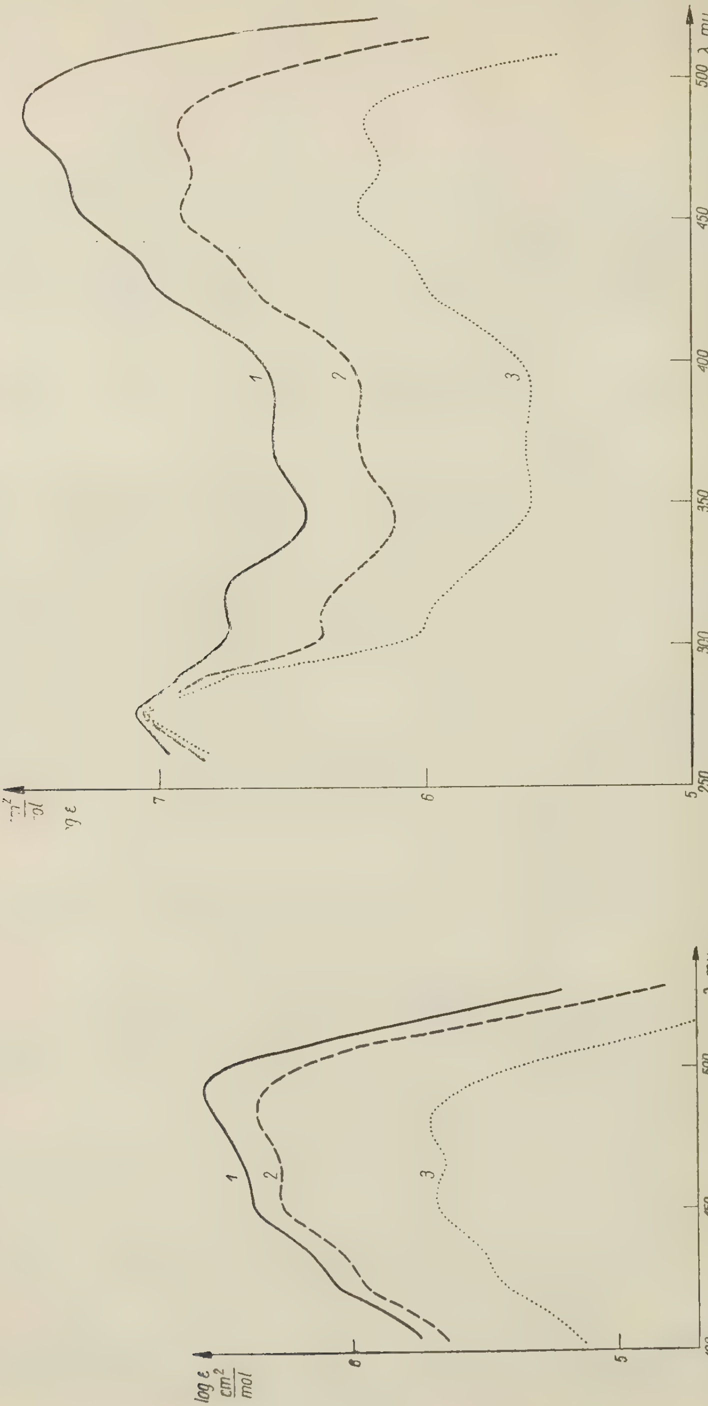

Fig. 1. Fluorescein absorption (a) yellow, b) red in methanol for following concentrations: 1) 6.02×10^{-7}, 2) 6.02—10^{-8}, 3) 6.02×10^{-9} mol/cm^3

spectra absorption bands may be seen distinctly with maxima at 485 mμ, 457 mμ, and 278 mμ, and traces of maxima at $\lambda = 430$ mμ, 375 mμ and 315 mμ. It is seen from the Figure, that absorption in the visible region and in the near ultraviolet sharply decreases, as the dye concentration increases. In the more distant ultraviolet, however, at $\lambda = 278$ mμ, the absorption remains approximately constant.

The decrease in intensity of the particular absorption bands is not identical, e.g., maxima at 315 mμ and 375 mμ vanish almost completely, while the 485 mμ maximum decreases more rapidly than that at 457 mμ. Although the localization of the absorption bands and their character are almost identical for both the initial forms of fluorescein, a certain difference may be observed in the course of vanishing of the absorption in the visible region: total absorption of red fluorescein diminishes more rapidly than that of yellow fluorescein.

Data on the mutual ratios of intensities in the maxima of the corresponding absorption bands for both fluorescein forms are given in Table I.

TABLE I

Intensity ratios in absorption bands maxima at 485 mμ and 457 mμ for various concentrations

Concentration of solution	6.02×10^{-7} mol./cm.^{3}	6.02×10^{-8} mol./cm.^{3}	6.02×10^{-9} mol./cm.^{3}
Yellow fluorescein	1.46	1.23	1.03
Red fluorescein	1.515	1.012	0.945

A similar course of absorption changes may be observed also for solutions with constant dye concentration but varying polarity. With polarity decreasing, the intensities of the main absorption bands decrease and so do the intensities of the bands 430 mμ, 375 mμ and 315 mμ (Figs. 2a and 2b). It is to be noted, however, that — similarly as in Fig. 1 — the intensity of the maximum at 485 mμ decreases more sharply than that at 457 mμ. The intensity of the maximum at 278 mμ remains constant within the limits of experimental error (intense absorption of CCl_4 appearing already in this region is responsible for considerable errors).

Data on the mutual ratios of intensities of maxima are listed in Table II.

TABLE II

Intensity ratios in absorption bands maxima at 485 mμ and 457 mμ for solutions of various polarity

Volume ratio CH_3OH/CCl_4	1 : 0	1 : 1	1 : 5	1 : 25	1 : 100
Yellow fluorescein (conc. $1.505 \times$ $\times 10^{-7}$ mol./cm.^{3})	1.09	0.999	0.914	0.83	0.819
Red fluorescein (conc. 6.02×10^{-8} mol./cm.^{3})	1.023	0.96	0.88	0.823	0.833

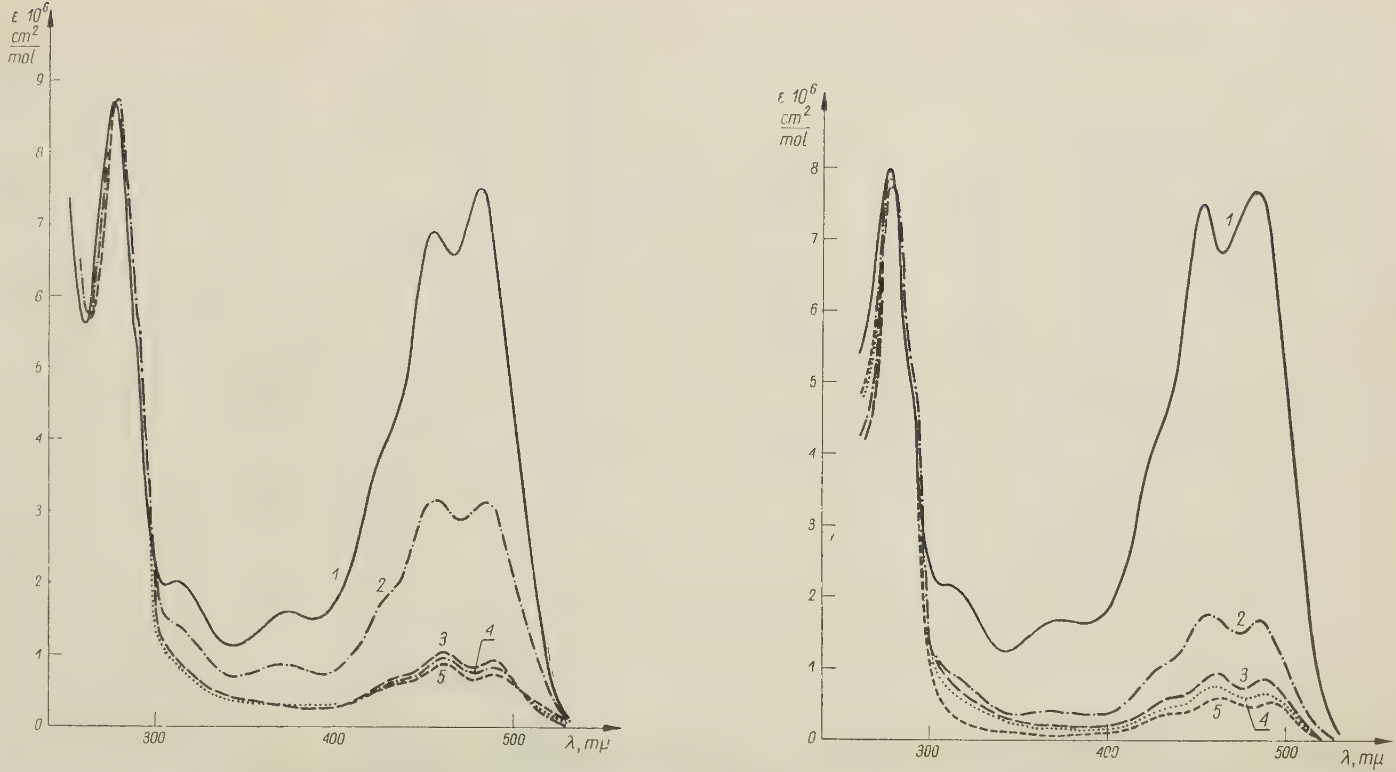

Fig. 2. Fluorescein absorption at constant concentration (a) yellow, conc. = 1.505×10^{-7} mol./cm³.; b) red, conc. – 6.02×10^{-8} mol./cm.³) in a mixture of methanol and CCl_4. Volume ratios of methanol and CCl_4 in the mixture: 1) 1 : 0, 2) 1 : 1, 3) 1 : 5, 4) 1 : 25, 5) 1 : 100

Note: Curve 4 in Fig. 2a should be denoted by a dotted line,

The same character of the course of vanishing of absorption bands (although the strength of the effect is different) may be observed in the case of fluorescein solutions of varying polarity but constant ratio of dye and methanol. These changes are shown in Fig. 3 for yellow fluorescein. It is to be noted that with increasing dye concentration or with decreasing of CCl_4 content in the solution, the absorption maxima in the visible spectrum shift towards the shortwave-lengths.

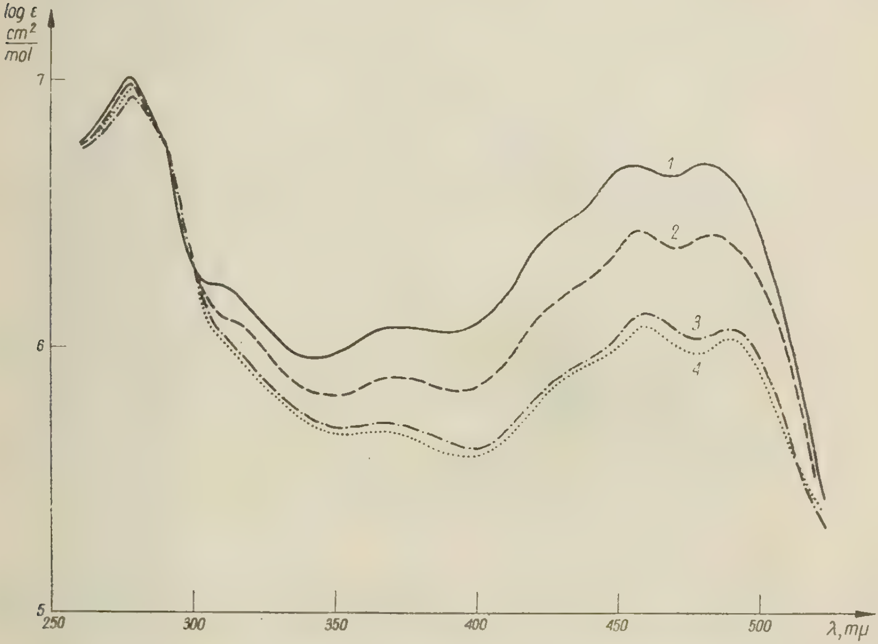

Fig. 3. Yellow fluorescein absorption at constant concentration 3.01×10^{-7} mol./cm.^{3} in the presence of methanol contained in the mixture methanol + CCl_4. Volume ratios of methanol and CCl_4 in the mixture: 1) 1 : 0 — fluorescein concentration in the solution 3.01×10^{-7}mol./cm.^{3}; 2) 1 : 1 — fluorescein concentration in the solution 1.505×10^{-7} mol./cm.^{3}; 3) 1 : 5 — fluorescein concentration in the solution 5.02×10^{-8} mol./cm.^{3}; 4) 1 : 25 — fluorescein concentration in the solution 1.16×10^{-8} mol./cm.^{3}

The behaviour of fluorescein in a neutral medium with a fairly high dipole moment (2.72 debye; for comparison: the dipole moment of methanol molecule amounts to 1.66 debye, whereas that of CCl_4 — 0.00 deb.) is different. As may be seen from Fig. 4, the absorption of fluorescein in acetone, as a rule, does not alter with increasing concentration; only the shape of the spectrum changes slightly.

By comparing different results reported above it may be easily seen that in media of relatively low polarity the behaviour of yellow as well as of red fluorescein at varying concentrations and at constant polarity of the solution is similar to that in a solution with varying polarity and at constant dye concentration. Let us add

that the changes of fluorescein absorption in the visible spectrum and in the near ultraviolet are not due to changes of concentration of the solutions, since no sediment of dye molecules precipitated from the solution was observed. This is corroborated by the fact that the value of the molar extinction coefficient in the ultraviolet remains unchanged (Figs. 1 and 2). Comparing our results with those of other authors ([2], [3], [6]—[8]) we may conclude that the absorption in the visible spec-

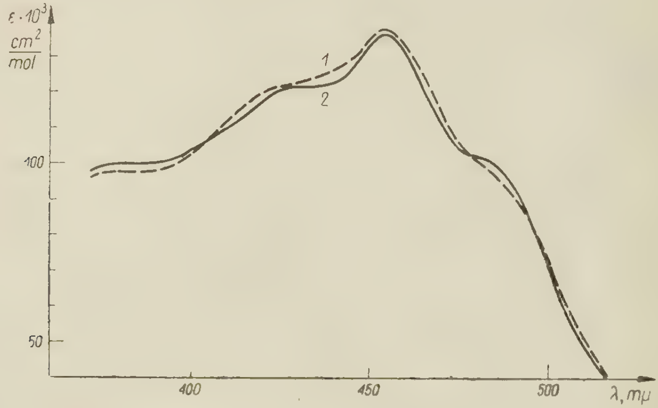

Fig. 4. Yellow fluorescein absorption in acetone for concentrations 1) 3.01×10^{-7} mol./cm.^{3} and 2) 3.01×10^{-8} mol./cm.^{3}

trum is due to the simultaneous occurrence in the same solution of cations as well as univalent and bivalent anions (the absorption bands are 430 mμ, 457 mμ 485 mμ, respectively).

If, according to Zanker [4], univalent anions would be responsible for both the 457 mμ and 485 mμ bands, then the course of their vanishing should be identical. But, as may be seen from Figs. 1 and 2, it is not so. Neither can univalent anions and their associates be held responsible for this discrepancy, because, as it follows from our further measurements and from [7], the rise of temperature does not lead to changes in the spectrum, characteristic for mixtures of monomer and their associates (cf., e.g., [10], [9], [2], [3]). Thus, it must be assumed that the 457 mμ band belongs to the univalent, while the 485 mμ band — to the bivalent anion in agreement with [1], [2], [7], [8], [3].

Summing up, we can state that in some neutral solutions, e.g. in methanol, fluorescein appears mainly as anions and neutral molecules. The chemical equilibrium between both forms depends on the dye concentration, polarity, temperature and pH [7] of the solution and, moreover, on the original modification of fluorescein. We may thus infer that in neutral and non-polar solvents the main absorption band of fluorescein should vanish completely. Some traces of these bands observed by Zanker [4] and by Bączyński, Czajkowski and Trawiński [8] are due to the fact

that dioxane molecules as well as those of the monomer of methyl metacrylate used by them as solvents, have still, although a low, dipole moment.

Our thanks are due to Professor A. Jabłoński for his interest in this paper.

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БЮЛЛЕТЕНЬ ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

ТОМ IX Резюме статей ВЫПУСК 6

Г. МИЛИЦЕР-ГРУЖЕВСКАЯ, **ГРАНИЧНОЕ СВОЙСТВО ОСОБОГО ПОТЕНЦИАЛА ПРОСТОГО СЛОЯ ПАРАБОЛИЧЕСКОЙ СИСТЕМЫ УРАВНЕНИЙ** . стр. 429—433

В работе доказывается, что границей особого потенциала простого слоя для параболической системы уравнений является особый потенциал простого слоя с граничной плотностью эллиптической системы граничных уравнений.

Автор рассматривает n - размер эвклидова пространства E_n не больший M - ряда системы уравнений ($n \leqslant M$). С этой целью конструируется особое основное решение системы параболических уравнений при соединении двух разных методов, а именно: методов С. Д. Эйдельмана и В. Погожельского [1].

А. БУРАЧЕВСКИЙ, **ДЕТЕРМИНАНТНЫЕ СИСТЕМЫ ДЛЯ ОБОБЩЕННЫХ ОПЕРАТОРОВ ФРЕДГОЛЬМА** . . . . стр. 435—440

В первых годах текущего столетия Фредгольм первый дал начало детерминантной теории линейных интегральных уравнений. Затем в 1953 г. Лежанский [2] обобщил эту теорию на уравнения с оператором типа Фредгольма, действующим в пространстве Банаха. Р. Сикорский [3], [4] развил ее и усовершенствовал, вводя понятие детерминантной системы, состоящей из определителя и миноров всех порядков.

Главной целью предлагаемой автором работы есть обобщение детерминантной теории на более широкий класс операторов, так называемых обобщенных операторов Фредгольма. Полученные формулы решения уравнений аналогичны формулам в случае оператора Фредгольма.

Работа состоит из двух частей: алгебраической и аналитической. В алгебраической части указаны свойства обобщенных операторов Фредгольма и детерминантных систем этих операторов в линейных пространствах без нормы.

В аналитической части автор занимается детерминантными системами для обобщенных операторов Фредгольма типа $S+T$, действующих в пространствах Банаха, где S — квази-обратимый и T — оператор Лежанского [2]. Здесь тоже указан метод, по которому можно получить детерминантную систему и тем самым эффективно решить уравнение.

В частности этот метод можно применить к решению линейных, сингулярных интегральных уравнений.

Р. ТАБЕРСКИЙ, **О КЛАССАХ Λ_M^α И λ_M^α 2π-ПЕРИОДИЧЕСКИХ ФУНКЦИЙ** стр. 441—444

В работе сначала приводятся некоторые свойства интегральных классов Липшица Λ_M^α и λ_M^α $(0 < \alpha \leqslant 1)$, а затем доказывается абсолютная сходимость ряда Фурье функции $f \in \Lambda_M^1$ при условии (7).

В. МАТУШЕВСКАЯ, **НЕКОТОРЫЕ СВОЙСТВА φ-ФУНКЦИИ** . . . стр. 445—450

В настоящей работе автор рассматривает некоторые свойства показателей s_φ, σ_φ, приведенных в работе [5].

Кроме того, автор обсуждает l-равносильность φ-функции с локально аналитическими функциями.

С. БАЛЬЦЕЖИК, А. БЯЛЫНИЦКИЙ-БИРУЛЯ и Г. ЛОСЬ, **О ПРЯМЫХ СЛАГАЕМЫХ ПОЛНОЙ ПРЯМОЙ СУММЫ ГРУПП ПЕРВОГО РАНГА** . . . стр. 451—454

Все рассматриваемые в работе группы являются абелевыми. Пусть H — является группой без кручения первого ранга и пусть $\mathfrak{a} = (a_1, a_2, \ldots)$ является типом группы H. Тогда через $\mathfrak{a}^0$ определяем тип $(b_1, b_2, \ldots)$, где $b_i = a_i$, если $a_i = \infty$, и $b_i = 0$, если $a_i \neq \infty$, а H^0 обозначает группу первого ранга типа $\mathfrak{a}^0$. В работе дается доказательство следующей теоремы:

Теорема 1. *Пусть $\{H_t\}_{t \in T}$ — семейство групп без кручения первого ранга, удовлетворяющих следующим условиям:*

а) *мощность множества T — меры* 0;

б) *H_α не является полной для каждого $\alpha \in T$,*

с) *для всяких элементов α, $\beta \in T$ H_α либо изоморфно H_β либо H_α^0 не изоморфно ни одной подгруппе группы H_β.*

В таком случае каждое прямое слагаемое полной прямой суммы групп $\{H_t\}_{t \in T}$ изоморфно полной прямой сумме групп $\{H_t\}_{t \in T_1}$, где T_1 — подмножество множества T.

Доказательство состоит в сведéнии исследования прямых слагаемых полных прямых сумм к исследованию прямых слагаемых дискретных сумм, для которых теорема аналогичная Теореме 1 известна [3]. Такое сведение получается путем рассмотрения групп гомоморфизмов полной прямой суммы групп $\{H_t\}_{t \in T}$ в полную прямую сумму групп $\{H_t\}_{t \in T_0}$, где T_0 является соответственно подобранным подмножеством множества T.

Е. РАДЕЦКИЙ, **О МОДИФИЦИРОВАННЫХ ПОЛИНОМАХ ЛАНДАУ** стр. 455—456

В работе приводится несколько результатов, касающихся аппроксимации непрерывных функций модифицированными полиномами Ландау.

М. К. ФОРТ младш., **ДОПОЛНЕНИЯ ОГРАНИЧЕННЫХ ОБЛАСТЕЙ ЭВКЛИДОВА ПРОСТРАНСТВА** . . . . стр. 457—460

В работе доказывается следующая теорема:

Мощность множеств компонент дополнений двух гомеоморфных ограниченных областей эвклидова пространства E^n одинакова.

Аналогичная теорема справедлива для сферы S^n. Отсюда вытекает, между прочим, что *два компактные множества* K_1 и $K_2 \subset S^n$ *размерности* 0 *гомеоморфны и одинаково расположены в* S^n *тогда и только тогда, когда их дополнения гомеоморфны.*

Р. ЖЕЛЯЗНЫ и А. КУШЕЛЛЬ, **ОСОБАЯ МОДЕЛЬ В ДВУГРУППОВОЙ ТЕОРИИ ТРАНСПОРТА НЕЙТРОНОВ** . . . . стр. 461—466

В работе продискутирована особая модель двугрупповой теории транспорта нейтронов, охарактеризованная через $l_1 = l_2$. Эта модель дает возможность точного решения многих многогрупповых проблем.

Для иллюстрации продискутировано решение проблемы Мильна для полупространства совместно с обсуждением проблемы длины экстраполяции, а также продискутирована критическая задача для пластины. Дискуссия опирается на применении процедуры Кейса.

В. ГАРЧИНСКИЙ, **О НЕКОТОРОМ ПРЕДСТАВЛЕНИИ АМПЛИТУДЫ РАССЕЯНИЯ В ТЕОРИИ ВОЗМУЩЕНИЙ** . . . . .стр. 467—471

На основании α-представления для причинной функции распространения, получено общее выражение для вклада от произвольной диаграммы Фейнмана.

Такой вид амплитуды рассеяния является полезным при исследовании ее аналитических свойств.

Рассмотрен случай расходящейся диаграммы и учтено влияние регуляризации на общий вид вклада от нее.

В. ГАРЧИНСКИЙ, **О НЕКОТОРЫХ ТОПОЛОГИЧЕСКИХ СВОЙСТВАХ ДИАГРАММ ФЕЙНМАНА** . . . . .стр. 473—476

Показано, что с каждой связной диаграммой Фейнмана тесно связана некоторая матрица $e(b)$, которая определяется заданием направлений на внутренних ее линиях. Эта матрица естественно появляется при рассмотрении закона сохранения момента в узлах диаграммы.

Такого рода матрицы изучались в комбинаторной топологии и носят название матриц инцидентности.

Пользуясь методом индукции Боголюбова и Парасюка доказаны три свойства матрицы $e(b)$, имеющие прямое отношение к физическим применениям.

Полученные результаты позволяют применить метод индукции к исследованию аналитических свойств амплитуды рассеяния.

М. МИОНСЕК и М. СУФФЧИНСКИЙ, **ПРОСТРАНСТВЕННАЯ ГРУППА БЕЛОГО ОЛОВА. I. ТОЧКИ СИММЕТРИИ** .стр. 477—482

В работе дается описание решетки белого олова, причем избрана объемно-центрированная система. Описаны элементы пространственных групп для точек симметрии в зоне Бриллюэна.

Даются неприводимые представления простых групп для наиболее симметричных точек в зоне Бриллюэна. Подчеркивается подобие группы для структуры белого олова с пространственной группой алмаза.

М. МИОНСЕК и М. СУФФЧИНСКИЙ, **ПРОСТРАНСТВЕННАЯ ГРУППА БЕЛОГО ОЛОВА. II. ЛИНИИ И ПЛОСКОСТИ СИММЕТРИИ** . . . . стр. 483—487

Даются неприводимые представления простых и двойных групп для линии и плоскости в зоне Бриллюэна белого олова. Приводятся таблицы совместности для простых и двойных групп.

М. СУФФЧИНСКИЙ, **ПРОСТРАНСТВЕННАЯ ГРУППА БЕЛОГО ОЛОВА. III. ДВОЙНЫЕ ГРУППЫ** . . . . .стр. 489—495

Даются добавочные представления двойных групп для наиболее симметричных точек в зоне Бриллюэна белого олова. Приводятся таблицы совместности для дополнительных представлений.

И. ГЕЛЬДТ, **НЕОБРАТИМОЕ ФОТООБЕСЦВЕЧИВАНИЕ АКРИФЛАВИНА В ПОЛИВИНИЛОВОМ АЛКОГОЛЕ** . . . . .стр. 497—505

Предметом исследований была кинетика фотореакции обесцвечивания акрифлавина в полиалкоголе винила при комнатной температуре в зависимости от степени концентрации красителя. Приводятся схемы наиболее вероятных фотореакций, вызывающих обесцвечивание красителя, а также дается кинетика этих реакций.

В работе исчислены относительные выходы реакций обесцвечивания для разных концентраций красителя. Выход реакции понижается по мере роста концентрации красителя. Это явление автор объясняет наличием ассоциированных соединений а ифлавина и более коротким временем пребывания молекул красителя в метастабильном состоянии.

А. ГУТШЕ и Г. ВАЛЕРЫСЬ, **ВЛИЯНИЕ КОНЦЕНТРАЦИИ И ПОЛЯРНОСТИ СРЕДЫ НА СПЕКТР ПОГЛОЩЕНИЯ ФЛУОРОСЦЕИНА** . . . . . . .стр. 507—513

Исследовано влиян концентрации красителя и полярности среды на спектры поглощения желтого и красного видов флуоросцеина.

Констатировано на основании изменении в спектрах поглощения, что флуоросцеин в нейтральных средах и с небольшой полярностью выступает одновременно в ионной форме в качестве катиона либо одно- и двухвалентного аниона, а также в нейтральной форме, не абсорбирующей в видимой части спектра. Равновесие между этими двумя формами зависит, между прочим, от полярности среды и концентрации красителя.

TABLE DES MATIÈRES

Mathématique

Physique Théorique

Physique Expérimentale

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